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THE ELEMENTS

OF THE

DIFFERENTIAL CALCULUS

FOUNDED ON THE

METHOD OF RATES OR FLUXIONS

PART SECOND

BY

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PREFACE.

This small volume constitutes the *second part* of an elementary text-book on the Differential and Integral Calculus. The *first part* was printed by John Wiley & Son, New York, 1874.

The *third part* will contain chapters on the Evaluation of Indeterminate Forms; on Functions of two or more Independent Variables; and on the Application of the Differential Calculus to the Theory of Curves.

After the completion of the third part, it is our intention to incorporate the three parts in a single volume on the Differential Calculus; this will be published, and also a volume on the Integral Calculus, as soon as practicable.

J. M. R.
W. W. J.

U. S. NAVAL ACADEMY, *October*, 1875.

CHAPTER IV.

SUCCESSIVE DIFFERENTIALS AND DERIVATIVES.

XII.

THE DIFFERENTIATION OF FUNCTIONS OF THE TIME.

65. When a variable quantity is expressed as a function of t (the time elapsed from a given instant or *origin* of time), its *rate* has a definite value for each instant (Art. 11), and is simply a function of the time. The processes already established for the differentiation of functions of x apply likewise to those of t; thus,—

$$d\left(\frac{a}{x}\right) = -\frac{a\,dx}{x^2},$$

so likewise

$$d\left(\frac{a}{t}\right) = -\frac{a\,dt}{t^2}.$$

This expression for the rate is a function of no variable except t; because, when t denotes the elapsing time, dt is not arbitrary like dx (to which any value may be assigned), but is obviously constant. In computing the numerical value of a rate, dt is taken as unity; for the date of elapsing time increases at the rate of one second per second: it is, nevertheless, necessary to retain the symbol dt in Algebraic expressions to characterize them as *differential expressions* derived from functions of t. This necessity will be more apparent as we proceed.

SPACE DESCRIBED AND VELOCITY.

66. The application of the principles of the Calculus to functions of the time is of frequent occurrence in Mechanics. Thus, if s denote the space described in the time t by a moving body under the influence of known forces, s will be a function of t, and its derivative $\frac{ds}{dt}$ will also be a function of t. Since $dt = 1$, this derivative will be equal to the rate of the variable s, and hence to the velocity of the moving body (Art 2). Denoting the velocity by v, we have, therefore,

$$v = \frac{ds}{dt}.$$

67. To illustrate, let us take the well-known formula for a freely falling body,

$$s = \tfrac{1}{2}gt^2,$$

in which g denotes a constant whose value is determined by experiment, and t the time of falling. Taking the derivative, we obtain

$$v = \frac{ds}{dt} = gt,$$

the formula for the velocity of a freely falling body.

SECOND DIFFERENTIALS.

68. When the differential of a variable is not constant, it has its own rate of change or differential. This differential of the differential is called the *second differential* of the original variable. In other words, x denoting the variable, if dx is not constant, $d(dx)$ will have a value. The symbol d^2x (which may be read *d-second x*) is used as an abbreviation for $d(dx)$.

69. Thus, resuming the functions of t discussed in Art. 9 and Art. 10, if

$$x = t^2,$$

it was remarked that dx must be an increasing quantity, hence d^2x should be positive; accordingly, we find

$$dx = 2t\,dt, \text{ and therefore } d^2x = 2(dt)^2.$$

Again, if

$$x = \frac{a^2}{t},$$

it was remarked that dx should be negative and numerically decreasing; accordingly, we have

$$dx = -\frac{a^2\,dt}{t^2}.$$

Since dx is negative and decreasing numerically, its *rate* must be regarded as one of increase, and consequently we should expect d^2x to be positive; in fact, we find

$$d^2x = \frac{2a^2(dt)^2}{t^3}, \text{ a positive quantity.}$$

In each of the above second differentials, the factor $(dt)^2$ presents itself, as dt presented itself in the first differentials, and is retained for a similar reason, although its numerical value is unity.

VELOCITY AND ACCELERATION.

70. The increase per unit of time in the velocity of a moving body is called in Mechanics the *acceleration* of the motion, and the decrease per unit of time in the velocity is called the *retardation* of the motion. The term *acceleration* is, however, in general so used as to include retardation whenever the formulas for determining it give negative values.

It is evident therefore that acceleration corresponds to a *positive* value of the second differential (see Art. 68) of the space described, and retardation to a *negative* value.

71. The velocity v of a moving body being a function of t, $\frac{dv}{dt}$ is also a function of t; and, since dt is unity, this derivative will be the measure of the rate of v, or of the acceleration of the motion. Denoting acceleration by a, we have, since

$$dv = d\frac{ds}{dt} = \frac{d^2s}{dt},$$

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

$(dt)^2$ is usually written dt^2 when it appears in the denominator of a derivative.

Resuming the illustration of Art. 67, we have, when

$$s = \tfrac{1}{2}gt^2, \quad v = \frac{ds}{dt} = gt, \quad \therefore a = \frac{d^2s}{dt^2} = g.$$

This expression for a indicates that, in the case of a freely falling body, the acceleration or increase of velocity per second is constant, and equal to the velocity acquired at the end of the first second, or when $t = 1$.

COMPONENT VELOCITIES.

72. When the motion of a point in a plane is referred to co-ordinate axes, the abscissa and ordinate are regarded as functions of t, and the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$, which (Art. 66) denote the rates of these variables, are called the component or resolved velocities along, or in the direction of, the axes. Denoting these component velocities by v_x and v_y, we have

$$v_x = \frac{dx}{dt} \quad \text{and} \quad v_y = \frac{dy}{dt}.$$

Again, denoting by s the actual space described as measured from some fixed point of the path, s will likewise be a function of t, and the derivative $\frac{ds}{dt}$ will denote the actual velocity of the point. Now, if (the axes being rectangular) φ represents the angle between the *positive* direction on the axis of x, and the direction of the motion when s is *increasing*, we have (compare Equations 4, Art. 53)

$$dx = ds \cos\varphi \quad \text{and} \quad dy = ds \sin\varphi.$$

Hence,

$$\frac{dx}{dt} = \frac{ds}{dt}\cos\varphi \quad \text{and} \quad \frac{dy}{dt} = \frac{ds}{dt}\sin\varphi.$$

or,

$$v_x = v\cos\varphi \quad \text{and} \quad v_y = v\sin\varphi.$$

Squaring and adding,

$$v_x^2 + v_y^2 = v^2.$$

The last equation enables us to determine the actual velocity in the curve from the component velocities.

73. If we represent the accelerations of the resolved motions along the axes by a_x and a_y, we shall have, by Art. 71,

$$a_x = \frac{d^2x}{dt^2} \quad \text{and} \quad a_y = \frac{d^2y}{dt^2}.$$

a_x and a_y will be positive when the resolved motions are *accelerated in the positive directions of the corresponding axes*, that is, when they increase a positive resolved velocity, or numerically decrease a negative resolved velocity.

EXAMPLES XII.

1. The space in feet described in the time t by a point moving in a straight line is expressed by the formula

$$s = gt + 16t^2;$$

find the acceleration and the velocity at the end of $5\frac{1}{2}$ seconds; also the initial velocity.

2. If the space described in t seconds be expressed by the formula

$$s = 10 \log \frac{4}{4+t};$$

find the velocity at the end of one second; at the end of six seconds; and at the end of sixteen seconds.

3. Show that, in the case of a point moving as in Example 2, the acceleration is proportional to the square of the velocity.

4. If a point move in a fixed path so that

$$s = \sqrt{t},$$

show that the acceleration is negative and proportional to the cube of the velocity. Find the value of the acceleration at the end of one second; at the end of nine seconds.

5. If a point move in a straight line so that its distance from a fixed point of the line is $a \cos \frac{1}{2}t\pi$, express its velocity at any time. When will the velocity be zero?

6. If $s = ae^t + be^{-t}$, show that the acceleration at any point is equal to the space described.

7. Prove that, if the square of the space described is proportional to the time, the product of the space described and the velocity at any instant is constant.

8. If a point referred to rectangular co-ordinate axes move so that

$$x = a \cos t + b \quad \text{and} \quad y = a \sin t + c,$$

show that its velocity will be uniform. Find also the equation of the path described. (Eliminate t from the given equations.)

9. If a point move in the parabola $ay = x^2$ so that $x = ct$, find expressions for v_x and v_y, and for the actual velocity v. What can be inferred from the value of v_y when t is negative?

10. Find the expressions for a_x and a_y in Example 9.

11. The velocity of a point is inversely proportional to the square of its distance from a fixed point of the straight line in which it moves, the velocity being 2 feet per second when the distance is 6 inches; determine the acceleration at a given distance s from the fixed point. *Ans.* $-\frac{1}{2s^5}$ feet.

12. The velocity of a point moving in a straight line is m times its distance from a fixed point at the perpendicular distance a from the straight line; show that the acceleration at the distance x from the foot of the perpendicular is expressed by $m^2 x$.

XIII.

SUCCESSIVE DIFFERENTIALS.

74. The differential of a function of an independent variable x contains dx as a factor; and, unless dx is a constant,—in other words, if x is not a uniformly varying quantity,—the expression for the second differential is more complex than in the cases of the functions of t discussed in XII, where dt is a constant factor.

Thus, given

$$y = x^3, \quad \ldots \quad (1)$$

then

$$dy = 3x^2\,dx, \quad \ldots \quad (2)$$

$$d^2y = 6x\,(dx)^2 + 3x^2\,d^2x. \quad \ldots \quad (3)$$

Since the differential dy in Equation (2) is the product of the variable factors $3x^2$ and dx, its differential is obtained by the formula for the product; the differential of dx being written d^2x.

The differential of the second differential is called the *third differential* of the given function, and is denoted by d^3y. Differentiating Equation (3), we obtain, on reducing,

$$d^3y = 6\,(dx)^3 + 18x\,dx\,d^2x + 3x^2\,d^3x. \quad \ldots \quad (4)$$

EQUICRESCENT VARIABLES.

75. In many applications of the Calculus, the value of the differential of the independent variable is arbitrary. In such cases, the results of successive differentiation are simplified by making dx a constant quantity, which is equivalent to making x vary at a uniform rate. When this assumption is made, the independent variable is said to be *equicrescent.*

Thus, x being equicrescent, if

$$y = x^3,$$
$$dy = 3x^2\,dx,$$
$$d^2y = 6x\,(dx)^2,$$
$$d^3y = 6(dx)^3,$$
$$d^4y = 0.$$

The differentials succeeding the fourth will in this case likewise vanish. If, in the more general form of these differentials, given in Equations (3) and (4) of the preceding Article, we make $d^2x = 0$, $d^3x = 0$, &c., we obtain the simpler forms given above.

Again, if we have (x being equicrescent)

$$y = a^x,$$

we find

$$dy = \log a \,.\, a^x.dx,$$
$$d^2y = (\log a)^2.a^x\,(dx)^2,$$
$$d^3y = (\log a)^3.a^x\,(dx)^3,$$
$$\cdots\cdots\cdots\cdots$$

and, in general,

$$d^ny = (\log a)^n.a^x\,(dx)^n.$$

The last expression is the nth differential of a^x, obtained, it must be remembered, on the assumption that dx is constant. The assumption in this case simplifies the results, but does not cause any of the differentials to vanish.

EXAMPLES XIII.

1. Find the general value of $d^2(a^x)$.

2. Find the second differential of $\sin \theta$, and reduce it to the simpler form by assuming θ equicrescent.

3. Assuming x equicrescent, find the successive differentials of $Ax^4 + Bx^3 + Cx^2 + Dx + E$.

4. Making the same assumption, find several successive differentials of $\sqrt{x}$ and of $\frac{1}{x}$.

5. Under what conditions will the series of differentials of an algebraic function of x all vanish after the mth?

6. Find the series of successive differentials of $\log x$, x being equicrescent.

7. Find the successive differentials of $\sin x$ and of $\cos x$.

8. Find three successive differentials of $\tan \theta$.

XIV.

SUCCESSIVE DERIVATIVES.

76. Since $\frac{dy}{dx}$ is used to denote the derivative of y as a function of x, $\frac{d}{dx}$ may be prefixed to any function of x to denote the operation of taking the derivative.

The derivative being in general a function of the independent variable x, the derivative of a derivative may be taken. The result thus obtained is called the *second derivative* of the original function. The symbol for the second derivative of $f(x)$ is formed by affixing a second accent to that of the first derivative; thus, $f''(x)$. The second derivative of y is written, in accordance with the notation introduced above, thus,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right).$$

77. Since the value of $\frac{dy}{dx}$ is independent of the absolute value of dx, we may make the assumption of Art. 75 without affecting the value of the second derivative, which, indeed, is a function of no variable except x.

Assuming x to be equicrescent, we have, therefore,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{(dx)^2}=\frac{d^2y}{dx^2}$$

The last is the symbol generally used; d^2y here denoting the simplified value of the second differential taken with the above assumption, and this assumption is implied whenever the notation $\frac{d^2y}{dx^2}$ is used.* It is customary to omit the parenthesis in the denominator of a derivative; thus, $(dx)^2$ is written dx^2.

---

* Were dx not assumed constant, the expression for the second derivative of y would be

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{dx\,d^2y-dy\,d^2x}{dx^3}.$$

When d^2x is assumed to be zero, this expression reduces to $\frac{d^2y}{dx^2}$.

SECOND DERIVATIVES ILLUSTRATED BY CURVES.

78. If the curve whose equation is

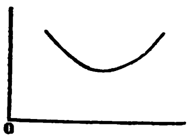
$$y = f(x)$$

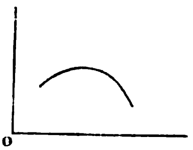
be constructed, we have seen (Art. 26) that

$$\frac{dy}{dx} = f'(x) = \tan\varphi,$$

φ being the inclination of the curve to the axis of x; $\tan\varphi$ affords therefore a geometrical illustration of the varying value of the first derivative.

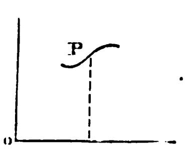
The curvature may be regarded as due to the variable character of $\tan\varphi$: when $\tan\varphi$ increases as x increases, and decreases as x decreases, the curve will be *concave* as viewed from above; thus, in the diagram, as we proceed toward the right, $\tan\varphi$ (at first negative) increases numerically throughout; accordingly, the curve is concave. On the other hand, if $\tan\varphi$ decreases as x increases,—that is, if it is a decreasing function of x,—the curve will be *convex* as viewed from above. [See diagram.]





Therefore, when $\frac{d}{dx}\tan\varphi$ or $\frac{d^2y}{dx^2}$ is *positive* between any two points of the curve, the portion included is concave as viewed from above; and if $\frac{d^2y}{dx^2}$ is *negative* between two points, the included portion of the curve is convex.

79. The curvature does not enable us to determine the value of $\frac{d^2y}{dx^2}$, but only its algebraic sign. If $\frac{d^2y}{dx^2}$ becomes zero, *and changes sign*, at any point of the curve, that point separates a concave from a convex portion of the curve, and is called a *point of contrary flexure*, or a *point of inflexion*. [See diagram.]



THIRD AND HIGHER DERIVATIVES.

80. The derivative of the second derivative is called the *third derivative* of the original function, and is denoted by

$f'''(x)$, or by $\dfrac{d^3y}{dx^3}$; the third derivative is likewise the second derivative of the first derivative. Thus we have the series of functions

$$f(x), f'(x), f''(x), f'''(x), f^{\text{iv}}(x) \;.\;.\;.\;.\; f^n(x)$$

each of which is the derivative of the preceding function; the general term of this series is read "function *n*th of *x*". If we have $y = f(x)$, we may therefore write

$$\frac{d^ny}{dx^n} = f^n(x).$$

EXAMPLES XIV.

1. Find the second derivative of $\sin x$, and determine the position of the concave and of the convex portions of the curve $y = \sin x$.

2. Show that the curve $y = \log x$ is everywhere convex.

3. Find the point of inflexion of the curve

$$y = 2x^3 - 3x^2 - 12x + 6.$$

4. Distinguish the concave from the convex portions of the curve $y = \sec x$.

5. Show that, if dx is constant, a concave portion of the curve corresponds to an acceleration of the upward motion of the generating point, and the reverse.

6. If $\dfrac{d^2y}{dx^2} = 0$, prove that y is a linear function of x.

7. Given the parabola $ay = x^2$, prove that $\dfrac{d^2y}{dx^2}$ is constant.

8. Find the *n*th derivative of $\log x$.

Ans. $(-1)^{n-1} . \dfrac{1.2 \ldots . (n-1)}{x^n}$.

9. Find the *n*th derivative of $\sin x$.

Ans. $\sin(x + \frac{1}{2}n\pi)$.

10. Show that, if one of the series of derivatives of a function takes the infinite form for a particular value of x, all succeeding derivatives likewise become infinite. [If u and v are functions of x, and $v = 0$ for the particular value of x, $\dfrac{u}{v}$ may be taken as the form specified.]

11. Find the successive derivatives of the function

$$\tfrac{1}{2}(e^x + e^{-x}).$$

12. If $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, find $f'(x)$ and $f''(x)$.

13. Find $\dfrac{d}{dx}[\log(e^x + e^{-x})]$; also $\dfrac{d^2}{dx^2}$ and $\dfrac{d^3}{dx^3}$ of the same function. (See Ex. 12.)

14. Find the nth derivative of xe^x. *Ans.* $ne^x + xe^x$.

15. Find the nth derivative of x^2e^x.

Ans. $[n(n-1) + 2nx + x^2]e^x$.

16. Find the nth derivative of xe^{2x}.

Ans. $[n2^{n-1} + 2^nx]e^{2x}$.

XV.

THE DERIVATIVES OF IMPLICIT FUNCTIONS.

81. When x and y are connected by an equation, either may be regarded as the independent variable of which the other is a function (Art. 14). When it is possible to put the function in the explicit form, its derivatives are found by the processes of Chapters II and III as explicit functions of the same independent variable. Thus, $f'(x)$, $f''(x)$, &c., can be found if the equation can be put in the form $y = f(x)$. But, if the equation cannot be solved for y, $f(x)$ denotes a function whose form is unknown; it is likewise impossible, in general, to determine the form of $f'(x)$, $f''(x)$, &c. The *values* of these derivatives may, however, be determined, but only in terms of x and y. Thus, suppose the relation between x and y to be expressed by

$$xy^2 + yx^2 - 2 = 0, \qquad . \quad . \quad . \quad (1)$$

we obtain, by differentiating,

$$y^2\,dx + 2xy\,dy + 2xy\,dx + x^2\,dy = 0;$$

whence

$$\frac{dy}{dx} = -\frac{y^2 + 2xy}{2xy + x^2}. \qquad . \quad . \quad . \quad (2)$$

82. In this case, by solving Equation (1) for y, it is possible to put it in the form $y = f(x)$; thus,

$$y = -\frac{x}{2} \pm \sqrt{\frac{x^2}{4} + \frac{2}{x}}. \qquad . \quad . \quad . \quad (3)$$

The value of $\frac{dy}{dx}$, or $f'(x)$, might therefore have been obtained directly in terms of x by the differentiation of (3).

Since there are two values of $f(x)$ in (3), there will in this case be two values of $f'(x)$. Equation (2) gives but one *expression* for $f'(x)$; but if the two values of y in Equation (3) were separately substituted in (2), we should obtain two values of $f'(x)$. There is, however, but a single value of $\frac{dy}{dx}$ for a given one of the sets of values of x and y which satisfy Equation (1); thus, $x = 1$ and $y = 1$ being values which satisfy Equation (1), the corresponding value of $\frac{dy}{dx}$ is -1.

83. To find the second derivative, we differentiate the expression obtained for the first derivative, and divide by dx; we shall thus obtain an expression involving not only x and y but $\frac{dy}{dx}$ also; substituting the value of the latter, we deduce an expression for $\frac{d^2y}{dx^2}$ involving only x and y. Thus, given the equation

$$\log(x+y) = x - y,$$

we obtain, by differentiating and reducing,

$$(x+y+1)\,dy + (1-x-y)\,dx = 0.$$

Whence

$$\frac{dy}{dx} = \frac{x+y-1}{x+y+1};$$

differentiating and dividing by dx,

$$\frac{d^2y}{dx^2} = \frac{(x+y+1)\left(1+\frac{dy}{dx}\right) - (x+y-1)\left(1+\frac{dy}{dx}\right)}{(x+y+1)^2};$$

substituting the value of $\frac{dy}{dx}$, we obtain

$$\frac{d^2y}{dx^2} = \frac{4(x+y)}{(x+y+1)^3}.$$

In like manner, the third derivative may be found.

DIFFERENTIAL EQUATIONS.

84. The equation obtained by differentiating a given equation between x and y is called the *first differential equation.* Thus, the first differential equation derived from

$$\log(x+y) = x - y \quad . \quad . \quad . \quad (1)$$

is

$$(x+y+1)\,dy + (1-x-y)\,dx = 0. \quad . \quad (2)$$

The former is called the *primitive* of the latter equation.

The first differential equation is, therefore, a relation between x, y, dx, and dy; or, since the ratio of the differentials can be expressed in terms of x and y, it may be regarded as a relation between x, y, and the derivative of y as a function of x, (or of x as a function of y.)

85. The *second differential equation* from a given primitive is derived by differentiating the first differential equation; thus, the second differential equation from

$$\log (x+y) = x - y$$

is

$$(1+x+y)\,d^2y + (1-x-y)\,d^2x + (dy)^2 - (dx)^2 = 0. \qquad (3)$$

It is obvious that each term of such an equation must necessarily contain either a second differential, or the product of two first differentials. A second differential equation expresses a relation between x, y, dx, dy, d^2x, and d^2y.

86. The value of the second derivative may readily be deduced from the second differential equation whenever (as is generally the case) the independent variable can be made equicrescent. Thus, dx being constant, Equation (3) becomes

$$(1+x+y)\,d^2y + (dy)^2 - (dx)^2 = 0,$$

dividing by $(dx)^2$, we obtain

$$(1+x+y)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - 1 = 0. \quad . \quad . \quad (4)$$

Eliminating $\frac{dy}{dx}$ by means of Equation (2), and reducing, we derive

$$\frac{d^2y}{dx^2} = \frac{4(x+y)}{(x+y+1)^3};$$

which is identical with the result obtained in Art. 83.

87. In a similar manner, differential equations of the third and higher orders may be formed; the *order of a differential equation* being that of the highest differential contained in it.

The remark (Art. 85) with reference to second differential equations may be generalized thus: *If we replace the indices of the differentials by exponents, the result will be homogeneous with respect to differential factors.* A differential equation not having this property would be without meaning.

DERIVATIVES OF INVERSE FUNCTIONS.

88. The relation implied by an equation involving x and y may be expressed in either of the two forms

$$y = f(x), \quad \text{or} \quad x = \varphi(y).$$

Even when the form of one or both of these functions is unknown, or cannot be expressed by a combination of known symbols, each is called the inverse of the other, as in Art. 57, where the relation between the first derivatives was found.

The successive derivatives of φ may be expressed in terms of those of f as follows:

Taking successive differentials of the equation

$$y = f(x),$$

x not being equicrescent, we derive

$$dy = f'(x)\,dx, \quad \ldots \ldots \ldots \quad (1)$$

$$d^2y = f''(x)(dx)^2 + f'(x)\,d^2x, \quad \ldots \ldots \quad (2)$$

$$d^3y = f'''(x)(dx)^3 + 3f''(x)\,dx\,d^2x + f'(x)\,d^3x \quad . \quad (3)$$

Now, making y equicrescent, the derivatives of φ will be

$$\frac{dx}{dy},\ \frac{d^2x}{dy^2},\ \frac{d^3x}{dy^3},\ \&c.$$

Hence, we deduce from (1)

$$\varphi'(y) = \frac{dx}{dy} = \frac{1}{f'(x)}, \quad \ldots \quad (4)$$

and from (2), making $d^2y = 0$,

$$\varphi''(y) = \frac{d^2x}{dy^2} = -\frac{f''(x)}{f'(x)}\left(\frac{dx}{dy}\right)^2;$$

or, substituting the value of $\frac{dx}{dy}$,

$$\varphi''(y) = -\frac{f''(x)}{[f'(x)]^3}. \quad \ldots \quad (5)$$

We derive from (3), by making $d^3y = 0$,

$$\varphi'''(y) = \frac{d^3x}{dy^3} = -\frac{f'''(x)}{f'(x)}\left(\frac{dx}{dy}\right)^3 - 3\frac{f''(x)}{f(x)}\frac{dx}{dy}\frac{d^2x}{dy^2};$$

or, substituting the values of $\frac{dx}{dy}$ and $\frac{d^2x}{dy^2}$ from (4) and (5),

$$\varphi'''(y) = \frac{3[f''(x)]^2 - f'''(x)f'(x)}{[f'(x)]^5}. \quad \ldots \quad (6)$$

In a similar manner, formulas for the higher derivatives might be deduced. Formulas for the derivatives of f in terms of φ would be, of course, identical in form with those given above.

EXAMPLES XV.

1. Given $a^2y^2 + b^2x^2 = a^2b^2$, find $\frac{dy}{dx}$ from the first differential equation, and show that the result is identical with the value found from the explicit form.

2. Given $x^2(y-1)+y^2(x+1)=1$, to find a general expression for $\frac{dy}{dx}$, and also its numerical values when $y=2$. [Deduce the corresponding values of x from the original equation.]

3. Given $x^y = y^x$, find a general expression for $\frac{dy}{dx}$, and simplify the result by means of the given equation. Show that, if x and y are equal, but not equal to e, $\frac{dy}{dx}=1$.

4. Given $e^x = y^3 - xy$, to express $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ by means of algebraic functions. [Eliminate e^x by the given equation.]

5. Given $e^x + x = e^y - y$; find $\frac{d^2y}{dx^2}$ by taking the derivative of $\frac{dy}{dx}$; also, find each of the second derivatives from the second differential equation.

6. Write the first and second differential equations of $y^3 - 3axy + x^3 = 0$; and from them derive $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$.

The differential equations are

$$(y^2 - ax)\,dy + (x^2 - ay)\,dx = 0,$$

and

$$(y^2-ax)\,d^2y + (x^2-ay)\,d^2x + 2y\,(dy)^2 - 2a\,dx\,dy + 2x\,(dx)^2 = 0.$$

To derive $\frac{d^2y}{dx^2}$, make $d^2x = 0$, and eliminate dy; we thus obtain

$$(y^2 - ax)\,d^2y + 2\left[y\left(\frac{ay - x^2}{y^2 - ax}\right)^2 - a\,\frac{ay - x^2}{y^2 - ax} + x\right](dx)^2 = 0.$$

Whence

$$(y^2 - ax)\,d^2y + 2xy\,\frac{x^3 - 3axy + y^3 + a^3}{(y^2 - ax)^2}\,(dx)^2 = 0,$$

and, by substitution from the primitive equation, we derive

$$\frac{d^2y}{dx^2} = -\frac{2a^3xy}{(y^2 - ax)^3},$$

the simplest expression for $\frac{d^2y}{dx^2}$ in terms of x and y.

7. Find from $Ax + By + C = 0$ the second differential equation, and interpret the result.

8. Derive $\dfrac{d^2x}{dy^2}$ from the second differential equation obtained from $y = x^2$, and show that the result agrees with that obtained by treating x as an explicit function.

9. If $y = ax^2 + bx$, x being equicrescent, show that

$$x^2d^2y - 2x\,dy\,dx + 2y(dx)^2 = 0.$$

[Eliminate a and b by means of the primitive and first and second differential equations.]

10. Find a similar equation, y being equicrescent, and show, by means of the relations between the derivatives given in Art. 88, that the result is equivalent to the differential equation given in Example 9.

11. If $y = f(x)$ and $x = \varphi(y)$, convert the equation

$$a^2\{f''(x)\}^2 = \{1 + [f'(x)]^2\}^3$$

into a relation between the derivatives of φ.

12. The functions $y = \log(x + \sqrt{x^2 + a^2})$ and $x = \dfrac{a}{2}\left[\dfrac{e^y}{a} - \dfrac{a}{e^y}\right]$ are each the inverse of the other; find the first and second derivatives of each, and show that Equation (5) of Art. 88 is satisfied by them. Show that this fact alone does not prove that these are inverse functions, and complete the proof by considering the values of the functions and their first derivatives when $x = 0$.

CHAPTER V.

MAXIMA AND MINIMA OF FUNCTIONS OF A SINGLE VARIABLE.

XVI.

89. If, while the independent variable increases continuously, a function dependent on it increase up to a certain value, and then decrease, this value of the function is said to be a *maximum* value; on the other hand, if it decrease to a certain value, and then increase, such a value of the function is said to be a *minimum* value.

Thus, $n + \sin x$ increases until x reaches the value $\frac{1}{2}\pi$, and then decreases; the value $n + 1$ of the function corresponding to this value of x is, therefore, a maximum. Similarly, a minimum value $n - 1$ is obtained by putting $x = \frac{3}{2}\pi$

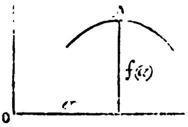
In accordance with the above definitions, if, while x increases through a particular value a, the function change from an increasing to a decreasing function, it will have a maximum value corresponding to $x = a$. On the other hand, if, while x increases through a, the function change from a decreasing to an increasing one, it will have a minimum value corresponding to $x = a$. In either case, *the derivative of the function changes sign* as x passes through the particular value a.

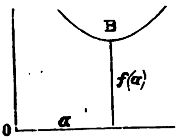
The case which most frequently presents itself is that in which the derivative passes through zero, when it changes sign. We shall, therefore, examine this case first, and afterward investigate (Art. 105) that in which the derivative changes sign without becoming zero.

90. If the curve whose equation is

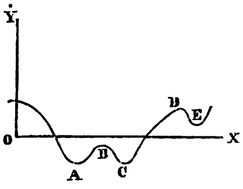
$$y = f(x)$$

have, at the point corresponding to the abscissa a, the form represented at A in the diagram, the ordinate of this point will be a maximum ordinate; but, if its form be that represented at B, the corresponding ordinate $f(a)$ will be a minimum. Accordingly, either of these points separates a part of the curve in which y is an increasing function from one in which y is a decreasing function; that is, a portion in which $f'(x)$ or $\tan\varphi$ is positive from one in which this quantity is negative. In either case, $f'(a)$ (the value of $\tan\varphi$ corresponding to $x = a$) is zero. If we regard x as increasing through the value a, then, in the case of the maximum, $f'(x)$ changes sign in the order $+\ 0\ -$, while, in the case of the minimum, the order is $-\ 0\ +$. Conversely, if $f'(a) = 0$, and $f'(x)$ change sign as x increases through the value a, $f(a)$ will be a maximum or a minimum value of $f(x)$, according as the change of sign takes place in the order $+\ 0\ -$ or $-\ 0\ +$.





Functions frequently have several maxima and minima; thus, in the diagram, a maximum value of the ordinate occurs at B and at D, and a minimum value at A, at C, and at E. It is to be noticed that an ordinate of either class may be either positive or negative, and that a minimum ordinate may be greater than a maximum ordinate; *e. g.*, the ordinate of E is greater than that of B.



91. To find the abscissas of points like A and B, we put

$$f'(x) = 0,$$

and solve for x. It is obvious that this equation will, in general, have several roots, which will be abscissas of as many points in which the curve is parallel to the axis of x; but one or more of these roots may correspond, as will be explained hereafter, to ordinates, or values of $f(x)$, which are neither maxima nor minima.

The existence of the maximum or minimum value required in each of the following examples is obvious at the outset.

92. Let it be required *to divide a number into two such parts that the square of one part multiplied by the cube of the other shall give the greatest possible product.*

Denote the given number by a, and the part to be squared by x; then

$$x^2(a-x)^3$$

will be the function whose maximum value is required. It is evident that such a maximum exists; for if $x = 0$ the product is zero, and when $x = a$ the product is again zero, while for intermediate values of x it is positive.

Taking the derivative of this function, the equation $f'(x) = 0$ becomes for this case

$$2x(a-x)^3 - 3x^2(a-x)^2 = 0,$$

or

$$x(a-x)^2(2a-2x-3x) = 0,$$

that is

$$x(a-x)^2(2a-5x) = 0.$$

0 and a are roots of this equation; but, as we are in search of a value of the function corresponding to an intermediate value of x, we put

$$2a - 5x = 0,$$

and obtain

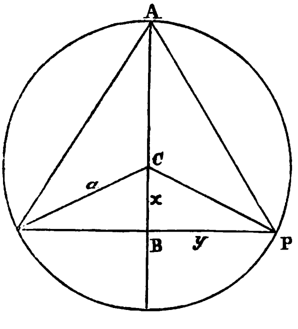
$$x = \tfrac{2}{5}a.$$

The corresponding value of the function $[\frac{108}{3125}a^5]$ is the maximum sought.

GEOMETRICAL PROBLEMS.

93. As an illustration, let the following problem be proposed: *Of all right cones inscribed in a given sphere, to determine the one having the greatest volume.*

Any point A of the surface of the sphere being taken as the apex of the cone, let the diagram represent a great circle of the sphere passing through the fixed point A. The position of the point P on the circumference of the circle is to be regarded as *variable*, and it is required to determine for what position of this point the volume of the cone is a maximum.



a, x, and y being defined by the diagram, we have, as an expression for the volume,

$$\tfrac{1}{3}\pi y^2(a+x).$$

Taking x as the independent variable, we put for y^2 its value in terms of x, and obtain

$$\tfrac{1}{3}\pi(a+x)(a^2-x^2).$$

The volumes of all the possible inscribed cones are represented by the values of this function while x varies from $-a$ to $+a$. As x increases from $-a$ to $+a$, the value of the function starts from zero and returns to zero, remaining positive within this interval; at least one maximum value exists therefore corresponding to an intermediate value of x.

The equation $f'(x)=0$ becomes in this case,

$$\tfrac{1}{3}\pi(a^2-2ax-3x^2)=0,$$

or

$$(a+x)(a-3x)=0,$$

an equation satisfied by

$$x=-a \quad\text{or}\quad x=\tfrac{1}{3}a.$$

The root $x=\frac{1}{3}a$, being between the required limits, evidently gives the maximum value of the volume sought, which is thus determined as that cone of which the altitude $x+a$ is two-thirds of the diameter of the sphere.

If the value of the maximum be required, it is readily obtained by substituting the corresponding value of x. In this case, it is

$$\tfrac{1}{3}\pi\,.\,\tfrac{4}{3}a\,.\,\tfrac{8}{9}a^2=\tfrac{32}{81}\pi a^3.$$

94. The only condition to be observed in the selection of the independent variable x is that the magnitude under consideration shall really be *determinate* for a fixed value of x. Thus, in the above example, a value of x determines without ambiguity a certain one of the series of cones. On the other hand, a value of y does not determine without ambiguity one of the series of cones; therefore, y should not be taken as the independent variable.

EXAMPLES XVI.

1. Find the sides of the largest rectangle that can be inscribed in a semicircle of radius a.

The sides are $a\sqrt{2}$ and $\frac{1}{2}a\sqrt{2}$.

2. Find the side of the smallest square that can be inscribed in a given square.

(Take as the independent variable the distance between the angles of the two squares.)

3. Find the sides of the maximum rectangle that can be circumscribed about a given rectangle $(a\,b)$.

The sides are equal.

4. A boatman 3 miles out at sea wishes to reach in the shortest possible time a point on the beach 5 miles from the nearest point of the shore; he can pull at the rate of 4 miles an hour, but can walk at the rate of 5 miles an hour; find the point at which he must land. (Express the whole time in terms of the independent variable.)

He must land one mile from the point to be reached.

5. If a square piece of sheet-lead whose side is a have a square cut out at each corner, find the side of the latter square in order that the remainder may form a vessel of maximum capacity.

The side of the square is $\frac{1}{6}a$.

6. A tinsmith was ordered to make an open cylindrical vessel of given volume, which should be as light as possible; find the ratio between the height and the radius of the base.

The height equals the radius of the base.

7. What should be the ratio between the diameter of the base and the height of cylindrical fruit-cans in order that the amount of tin used in constructing them may be the least possible?

8. Find the maximum cone of given slant height (a).

The radius of the base is $a\sqrt{\frac{2}{3}}$.

9. Find the maximum right cone of given convex surface.

Let x denote the height, and r the radius of the base; then

$$x = r\sqrt{2}.$$

10. A given weight is to be raised by means of a lever weighing n pounds per linear inch, which has its fulcrum at one end, and at a fixed distance a from the point of suspension of the weight w; find the length of the lever in order that the power required to raise the weight may be a minimum.

11. A rectangular court is to be built so as to contain a given area, and a wall already constructed is available for one of the sides; find its dimensions so that the least expense may be incurred.

The side parallel to the wall is double each of the others.

12. A Norman window consists of a rectangle surmounted by a semicircle. Given the perimeter, required the height and breadth of the window when the quantity of light admitted is a maximum.

The radius of the semicircle is equal to the height of the rectangle.

13. Prove that the rectangle with given perimeter and maximum area is a square; also that the rectangle with given area and minimum perimeter is a square.

14. A sphere has its centre in the surface of a given sphere; what must be its radius in order that the area of the surface intercepted by the given sphere may be a maximum?

(The area of a zone is measured by the product of the circumference of a great circle into the altitude of the zone.)

15. Find the point on the line joining the centres of two given spheres from which the greatest amount of spherical surface is visible.

XVII.

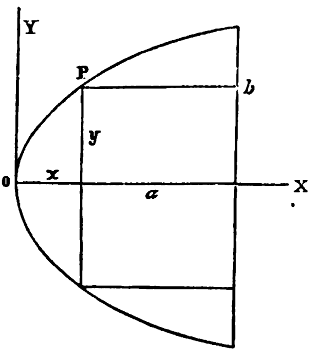
THE APPLICATION OF CO-ORDINATE GEOMETRY TO PROBLEMS OF MAXIMA AND MINIMA.

95. The relations between the variables are, in many instances, most conveniently determined by the known equations of curves referred to co-ordinate axes. As an example, let it be required *to determine the maximum rectangle inscribed in a parabolic segment whose base is* 2b *and altitude* a.

The origin of co-ordinates being taken at the vertex of the parabola, the relation between the co-ordinates of any point of the curve is of the form

$$y^2 = mx,$$

m being a constant. Since the point (a, b) is a point of the curve, we have the condition



$$b^2 = ma;$$

eliminating m, the equation of the curve is

$$y^2 = \frac{b^2}{a} x.$$

The rectangle of which the maximum is required is expressed by

$$2y\,(a - x);$$

substituting the value of y from the equation of the curve, the expression becomes

$$\frac{2b}{\sqrt{a}} \sqrt{x}\,(a - x).$$

Hence, we must have

$$(a - x)\sqrt{x} = \text{a maximum},$$

and (since $f'(x) = 0$)

$$\frac{a - x}{2\sqrt{x}} - \sqrt{x} = 0;$$

whence

$$x = \tfrac{1}{3} a.$$

The altitude $(a - x)$ of the rectangle is, therefore, two-thirds that of the parabolic segment.

A constant factor like $\frac{2b}{\sqrt{a}}$ may always be omitted from the expression to be made a maximum; in fact, if such a factor were retained, it would disappear in the solution of the equation $f'(x) = 0$.

96. As a further illustration, let it be required *to determine the greatest cylinder that can be inscribed in a given segment of a paraboloid of revolution.*

The above figure may now be employed to represent a section of the paraboloid. The volume of the cylinder generated by the rectangle $y(a - x)$ is

$$\pi y^2(a - x);$$

eliminating y, we obtain

$$\pi \frac{b^2}{a}(ax - x^2).$$

Whence, we deduce

$$a - 2x = 0,$$

or

$$x = \tfrac{1}{2}a.$$

$(a - x)$, the altitude of the cylinder, is, therefore, one-half the altitude of the paraboloid.

EXAMPLES XVII.

1. Find the maximum cylinder that can be inscribed in a given ellipsoid.

2. Inscribe the maximum cone in a given paraboloid, the apex of the cone being at the middle point of the base of the paraboloid.

3. Through a point whose rectangular co-ordinates are a and b draw a line such that the triangle formed by this line and the co-ordinate axes shall be a minimum.

4. A high vertical wall is to be braced by a shore which must pass over a parallel wall $6\frac{3}{4}$ feet high, and 16 feet distant from the other; find the length of the shortest beam that can be used for this purpose. (Take as the independent variable the inclination of the beam to the horizon.)

5. From a given point on the axis of the parabola $y^2 = 4ax$, determine the shortest line to the curve.

6. Find the minimum isosceles triangle circumscribed about a parabolic segment.

The altitude of the triangle is four-thirds the altitude of the segment.

7. The top of a pedestal which sustains a statue a feet in height is b feet above the level of a man's eye; find his horizontal distance from the pedestal when the statue subtends the greatest angle.

8. Inscribe the greatest rectangle in an ellipse.

9. The equation of the path of a projectile being

$$y = x \tan \alpha - \frac{x^2}{4h \cos^2 \alpha},$$

find the value of x when y is a maximum; also the maximum value of y.

Ans. $x = h \sin 2\alpha$, and $y = h \sin^2 \alpha$.

10. Determine the cylinder of greatest convex surface that can be inscribed in a given cone.

The surface $= \frac{1}{2}\pi rh$.

11. Find the least isosceles triangle that can be described about a given ellipse, having its base parallel to the major axis.

The height is three times the minor semi-axis.

12. Inscribe the greatest parabolic segment in a given isosceles triangle.

The altitude of the segment is three-fourths that of the triangle.

XVIII.

THE SECOND DERIVATIVE EMPLOYED TO DISCRIMINATE BETWEEN MAXIMA AND MINIMA.

97. When it is not apparent from the nature of the problem whether the value of the function corresponding to one of the roots of the equation $f'(x) = 0$ is a maximum or a minimum, it is usually possible to distinguish between them by examining the second derivative. Reference to the diagram (Art. 90) is sufficient to show that a maximum ordinate of the curve $y = f(x)$ is reached when the first derivative becomes zero in a *convex* portion of the curve, and a minimum when it becomes zero in a *concave* portion of the curve; by Art. 78, the curve is convex if the second derivative is negative, and concave if it is positive. Thus, a root of $f'(x) = 0$ corresponds to a *maximum* when the second derivative is negative, and to a *minimum* when it is positive.

98. In fact, a negative value of $f''(a)$ shows that $f'(x)$ *decreases* through the value $f'(a)$, so that if $f'(a) = 0$, $f'(x)$ changes sign in the order $+ 0 -$, which (see Art. 90) indicates that $f(a)$ is a maximum value of $f(x)$; on the other hand, a positive value of $f''(a)$ shows that the change of sign takes place in the order $- 0 +$, and hence that the value $f(a)$ is a minimum value.

Thus, given

$$f(x) = x^3 - 3x^2 - 9x + 5,$$

$$f'(x) = 3x^2 - 6x - 9;$$

hence, putting

$$x^2 - 2x - 3 = 0,$$

we obtain

$$x = -1, \quad \text{or} \quad x = 3.$$

These are the only values of x which can render this function either a maximum or a minimum. In this case

$$f''(x) = 6x - 6;$$

when $x = -1$,

$$f''(x) = -12,$$

and hence this value of x corresponds to a *maximum* value of the function; but when $x = 3$,

$$f''(x) = 12,$$

3 is therefore a value of x which corresponds to a *minimum*.

The maximum value of the function is found by substitution to be 10, while the minimum is -17.

99. If $f'(x)$ contain a positive factor which cannot change sign, this factor may be omitted; since we can determine whether $f'(x)$ increases or decreases through zero by examining the sign of the derivative of the remaining factor. Thus, if

$$f(x) = \frac{x}{1+x^2}, \quad f'(x) = \frac{1-x^2}{(1+x^2)^2}.$$

Since $\frac{1}{(1+x^2)^2}$ is always positive, we have only to determine whether the factor $1-x^2$ changes sign. Denoting this factor by v, and putting $v = 0$, we have

$$x = \pm 1.$$

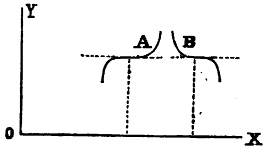
Now

$$\frac{dv}{dx} = -2x,$$

which is negative for $x = 1$ and positive for $x = -1$. These roots give, therefore, a maximum and a minimum value respectively of $f(x)$.

CASES IN WHICH THE FIRST AND SECOND DERIVATIVES VANISH.

100. If the first and second derivatives of $f(x)$ both vanish for a particular value a of x; that is, if $f'(a)$ and $f''(a)$ are both zero, we proceed to find $f'''(x)$, and to determine its value when $x = a$. Now, $f'''(x)$ has the same relation to $f'(x)$ that $f''(x)$ has to $f(x)$; it therefore follows that [$f''(a)$ being zero by hypothesis] $f'(a)$, although zero in value, is a *maximum* when $f'''(a)$ is negative, and a *minimum* when $f'''(a)$ is positive. As x passes through the value a, the order of signs of $f'(x)$ is, therefore, in the first case, $-\,0\,-$, and, in the second case, $+\,0\,+$. In both cases, $f'(x)$ *passes through zero without change of sign*, as x passes through the value a, and consequently $f(a)$ is *neither a maximum nor a minimum value of* $f(x)$.



101. If the curve $y = f(x)$ be constructed, the cases referred to above will be illustrated by points of inflexion at which the curve is parallel to the axis of x. (See Art. 79.)

The point A corresponds to the

case in which the order of signs of $f'(x)$ (or $\tan\varphi$) is $+\,0\,+$, and B to that in which it is $-\,0\,-$. In the first case, $f(x)$ (or y) continues to be an *increasing function*, and, in the second case, a *decreasing function*, while x passes through the value a.

EXAMPLES XVIII.

1. $f(x) = 2x^3 + 3x^2 - 36x + 12$; find the maximum and the minimum value of $f(x)$.

$$Ans. \begin{cases} x = 2 \text{ gives } f(x) = -32, \text{ a minimum.} \\ x = -3 \text{ gives } f(x) = 93, \text{ a maximum.} \end{cases}$$

2. Trace the curve $y = 2x^3 + 3x^2 - 36x + 12$, finding the minimum value of $\tan\varphi$ and the point at which it occurs.

The point is $(-\frac{1}{2}, 30\frac{1}{2})$.

3. $f(x) = 3x^5 - 125x^3 + 2160x$.

$$Ans. \begin{cases} x = -4 \text{ and } x = 3 \text{ give max. values of } f(x); \\ x = -3 \text{ and } x = 4 \text{ give min. values of } f(x). \end{cases}$$

4. $f(x) = \dfrac{1 - x + x^2}{1 + x - x^2}$; show that $x = \frac{1}{2}$ makes $f(x)$ a minimum. (See Art. 99.)

5. $f(x) = 3x^4 - 16x^3 - 6x^2 + 12$.

Ans. $x = 2 - \sqrt{5}$ gives a min., $x = 0$, a max., $x = 2 + \sqrt{5}$, a min.

6. $f(x) = \dfrac{x}{\log x}$.

$$f'(x) = \frac{\log x - 1}{(\log x)^2},$$

and

$$\log x - 1 = 0 \text{ gives } x = e.$$

It is unnecessary in this case to examine $f''(x)$; because as x passes through the value e, $f'(x)$ obviously changes sign in the order $-\,0\,+$; $f(x)$ is therefore a minimum for $x = e$. (See Art. 97.)

7. $f(x) = ae^{kx} + be^{-kx}$; show that $f(x)$ has a minimum value equal to $2\sqrt{ab}$.

8. $f(x) = \sin 2x - x$.

The roots of $f'(x) = 0$ are expressed by $x = n\pi \pm \frac{1}{6}\pi$, n being zero or any integer: the corresponding values of $f(x)$ are alternately maxima and minima, $x = \frac{1}{6}\pi$ giving a maximum.

9. $f(x) = x^x$. (See Example 6.)

10. $f(x) = x \sin x$.

The solution is given by the equation $\tan x = -x$: the first positive value of x is found in the second quadrant. Since $f(x)$ is zero when x is zero, and again when x equals π, being positive between these limits, this value of x gives a maximum.

11. $f(x) = b + c(x-a)^{\frac{5}{3}}$.

$$f'(x) = \tfrac{5}{3} c (x-a)^{\frac{2}{3}}.$$

Since $f'(x)$ cannot change sign, $f(x)$ has neither a maximum nor a minimum value.

12. $f(x) = ax^2 + bx + c$.

$f(x)$ has a maximum value when a is negative, and a minimum value when a is positive.

13. $f(x) = ax^3 + bx^2 + cx + d$.

The roots of $f'(x) = 0$ are real only when $b^2 - 3ac > 0$. This condition being fulfilled, if a is positive, it is obvious that, as x increases from $-\infty$ to $+\infty$, the function at first negative increases to the maximum, then decreases to the minimum, and afterward increases without limit. The value of x corresponding to the maximum is, therefore, less than that corresponding to the minimum.

14. $f(x) = x - \sin x$.

The function admits of neither a maximum nor a minimum value, since $f'(x)$ cannot change sign.

15. $f(x) = x(a+x)^2 (x-a)^3$.

$$\begin{aligned} f'(x) &= (a+x)^2 (x-a)^3 + 2x(a+x)(x-a)^3 + 3x(a+x)^2 (x-a)^2, \\ &= (a+x)(x-a)^2 [6x^2 + ax - a^2], \\ &= 6(x+a)(x-a)^2 (x - \tfrac{1}{3} a)(x + \tfrac{1}{2} a), \\ &\therefore\ (x+a)(x+\tfrac{1}{2} a)(x - \tfrac{1}{3} a)(x-a)^2 = 0, \end{aligned}$$

the roots of which are

$$-a,\ -\tfrac{1}{2} a,\ +\tfrac{1}{3} a,\ +a,\ +a.$$

When $x < -a$, $f'(x)$ is negative; when x increases through $-a$, $f'(x)$ passes through zero and becomes positive, again changing sign when $x = -\frac{1}{2} a$, and when $x = \frac{1}{3} a$, but *not* changing sign when $x = a$. The successive signs of the derivative are indicated below:

$$\begin{array}{lccccccccc} x = & & -a, & & -\tfrac{1}{2} a, & & \tfrac{1}{3} a, & & a, & \\ f'(x) & - & 0 & + & 0 & - & 0 & + & 0 & +. \end{array}$$

Therefore,

$x = -a$ gives a minimum,
$x = -\frac{1}{2}a$ gives a maximum,
$x = \frac{1}{2}a$ gives a minimum,
$x = a$ gives neither.

16. $f(x) = (x-1)^4(x+2)^3$.

$x = -\frac{5}{7}$ gives a max. $x = 1$ gives a min. $x = -2$ gives neither.

17. $f(x) = \dfrac{(x+3)^3}{(x+2)^2}$.

$$f'(x) = \frac{x(x+3)}{(x+2)^3}.$$

If we divide $f'(x)$ by the fraction $\dfrac{(x+3)^2}{(x+2)^4}$, which is positive for all values of x, we obtain $x(x+2)$; putting this result equal to v, we have (see Art. 99)

$$v = x(x+2).$$

The roots of $v = 0$ are

$$x = 0 \quad \text{and} \quad x = -2,$$

since

$$\frac{dv}{dx} = 2x + 2,$$

the first gives a minimum and the second a maximum.

18. $f(x) = \dfrac{x^2 - x + 1}{x^2 + x - 1}$.

$x = 0$ makes $f(x)$ a max., and $x = 2$, a min.

19. $f(x) = 2\cos x + \sin^2 x$.

$$f'(x) = 2\sin x(\cos x - 1).$$

Rejecting the factor $2(1-\cos x)$, which is always positive, we put

$$v = -\sin x.$$

When x passes through zero, v changes sign in the order $+\ 0\ -$, indicating a maximum value of $f(x)$. When x is equal to π, we have a minimum.

20. The illumination of a plane surface by a luminous point being directly as the cosine of the angle of incidence of the rays, and inversely as the square of its distance from the point; find the height at which a bracket-burner must be

placed, in order that a point on the floor of a room at the horizontal distance a from the burner may receive the greatest possible amount of illumination.

Show that the root $x = \frac{a}{\sqrt{2}}$ fulfils the conditions for a maximum, and $x = 0$, those for a minimum.

21. If $f'(x)$ contain a factor of the form $(x - a)^{2n+1}$, n being an integer, show that $f(a)$ is a maximum or a minimum; but, if the factor be of the form $(x - a)^{2n}$, that $f(a)$ is neither a maximum nor a minimum: also, that in the latter case the curve $y = f(x)$ has a point of inflexion at $x = a$.

22. If a rectangular piece of tin, whose sides are a and b, have a square cut out at each corner; find the side of the square in order that the remainder may form a box of maximum capacity.

$$\text{The side} = \frac{a + b - \sqrt{(a^2 - ab + b^2)}}{6}.$$

23. A conical vessel whose altitude is a and semi-vertical angle 30° is filled with liquid; find the radius of the sphere which, when placed in the vessel, will displace the greatest quantity of liquid.

The radius $= \frac{1}{2} a$.

24. $f(x) = \frac{a^2}{x} + \frac{b^2}{a - x}$.

The roots $x = \frac{a^2}{a + b}$ and $x = \frac{a^2}{a - b}$ give a min. and a max. if b is positive, but a max. and min. if b is negative.

XIX.

THE COMPLETE CRITERION.

102. Let us examine now the series of functions

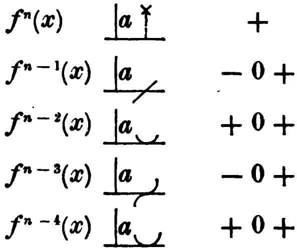
$$f(x),\ f'(x),\ f''(x),\ f'''(x),\ \ldots . f^{n-2}(x),\ f^{n-1}(x),\ f^{n}(x).$$

Let a denote one of the roots of $f'(x)=0$, and let $f''(a)$, $f'''(a), \ldots f^{n-2}(a)$, $f^{n-1}(a)$ vanish likewise; $f^{n}(a)$ being the first of the series of derivatives which does not vanish when $x=a$.

Let us first suppose $f^{n}(a)$ to be positive; it follows from this supposition that $f^{n-1}(x)$ is an increasing function when $x=a$, and, since $f^{n-1}(a)=0$, $f^{n-1}(x)$ changes sign in the order $-\ 0\ +$. Hence, $f^{n-2}(x)$, of which $f^{n-1}(x)$ is the derivative, has a *minimum* value when $x=a$. Now, $f^{n-2}(a)$ is zero by hypothesis, hence it is a *zero-minimum;* its sign is, therefore, positive both before and after $x=a$, as indicated by the signs $+\ 0\ +$; hence, $f^{n-3}(x)$ continues to be an increasing function as x passes through a.

Again, $f^{n-3}(a)$ is zero by hypothesis; hence, $f^{n-3}(x)$ changes sign in the order $-\ 0\ +$. Continuing this course of reasoning, it becomes evident that the functions of this series are *minima* and *increasing functions* alternately when $x=a$.

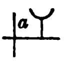
The diagram represents small portions of the curves $y=f^{n-1}(x)$, $y=f^{n-2}(x)$, &c., near the points for which $x=a$.



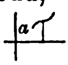
. . . .

. . . .

If, therefore, n is even,

$f(x)$  $f(a)$ a min.

If n is odd,

$f(x)$  a point of inflexion.

103. Now, since the original function $f(x)$ stands as the zero term of the series, if n is even, it will be one of the series f^{n}, f^{n-2}, &c., and will have a minimum value for $x=a$; but if n is odd, it will be one of the series f^{n-1}, f^{n-3}, &c., and the curve will have a point of inflexion, $f(a)$ being neither a maximum nor a minimum. If $f^{n}(a)$ is negative (the preceding derivatives still vanishing when $x=a$), it is proved in a precisely similar manner that the functions of the series are, for this value of x, alternately *maxima* and *decreasing functions; $f(a)$* being a maximum if n is even, but neither a maximum nor minimum if n is odd. The diagram is easily modified to illustrate this case.

104. The following example illustrates the application of the criterion.

$$f(x) = e^x + e^{-x} + 2\cos x,$$
$$f'(x) = e^x - e^{-x} - 2\sin x.$$

Zero is evidently a root of the equation $f'(x) = 0$. In this case

$$f''(x) = e^x + e^{-x} - 2\cos x \quad \therefore\ f''(0) = 0,$$
$$f'''(x) = e^x - e^{-x} + 2\sin x \quad \therefore\ f'''(0) = 0,$$
$$f^{\text{iv}}(x) = e^x + e^{-x} + 2\cos x \quad \therefore\ f^{\text{iv}}(0) = 4.$$

The fourth derivative being the first that does not vanish, and having a positive value, we conclude that $x = 0$ gives a minimum value of $f(x)$.

EXAMPLES XIX.

1. $f(x) = (x - 9)^5 (x - 8)^4$.

For $x = 8$, a max.

2. $f(x) = \log\cos x - \cos x$.

For $x = 0$, a max.

3. $f(x) = 4x^2 + \cos 2x - \frac{1}{2}(e^{2x} + e^{-2x})$.

For $x = 0$, a max.

4. Find the maximum and minimum ordinates of the curve $y = x^4 + 2x^3 - 12x^2 - 40x - 34$.

The concurrence of a maximum and a minimum ordinate at the point $(-2, -2)$, corresponding to a pair of equal roots of $f'(x) = 0$, gives a point like B in the diagram of Art. 100. There is a minimum ordinate corresponding to the other root of $f'(x) = 0$.

5. Show that the curve $y = x^6 - 6x^4 + 4x^3 + 9x^2 - 12x + 3$ has a minimum ordinate corresponding to $x = 1$.

6. $f(x) = \sin x\,(1 + \cos x)$.

For $x = \frac{\pi}{3}$, a max. For $x = -\frac{\pi}{3}$, a min. For $x = \pi$, neither.

7. $f(x) = (3 - x)e^{2x} - 4xe^x - x$. Is there a maximum or minimum corresponding to $x = 0$?

Neither.

XX.

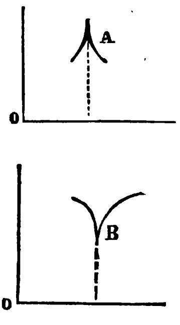
CASES IN WHICH THE FIRST DERIVATIVE BECOMES INFINITE.

105. The essential characteristic of a maximum or a minimum value of $f(x)$ is that $f'(x)$ changes sign as x passes through the corresponding value. The case in which it changes sign on passing through zero has been treated in the preceding sections of this chapter. $f'(x)$ may also change sign on passing through infinity; we now proceed to the consideration of the latter case.

Suppose we find that, for a value a of x, $f'(x)$ becomes infinite: it is then necessary, first, to ascertain whether $f(a)$ is infinite; and, secondly, whether $f(x)$ remains real while x passes through the value a.

Since a maximum value is preceded by increase and followed by decrease, the derivative must change sign in the order $+ \infty -$. On the other hand, in the case of a minimum the order must be $- \infty +$. If the derivative does not change sign, there is, of course, at this point neither a maximum nor a minimum.

106. If the curve $y = f(x)$ be constructed, a case in which the above conditions are fulfilled will be illustrated by a point similar to A or B in the diagram; the tangent at either of these points being common to the two branches of the curve. Such points are called *cusps*. In these cases, the common tangent is parallel to the axis of y.



107. Since $f''(x)$ becomes infinite for values of x which render $f'(x)$ infinite, it is necessary to distinguish maxima from minima by direct examination of the first derivative.

As an illustration, let us take

$$f(x) = (x - \quad {}^{\frac{2}{3}} + b,$$

whence

$$f'(x) = \tfrac{2}{3}(x-a)^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x-a}}.$$

$f'(x)$ cannot become zero, but is infinite for $x = a$. Recurring to $f(x)$, and putting a for x, we have

$$f(a) = b,$$

a finite quantity. Moreover, $f'(x)$ changes sign, as x increases through the value a, in the order $- \infty +$; therefore b is a minimum value of $f(x)$.

The conclusion in this case might have been anticipated, since $(x - a)^{\frac{2}{3}}$ cannot become negative.

108. Again, let us take

$$f(x) = (x - a)^{\frac{1}{4}},$$

whence

$$f'(x) = \tfrac{1}{4}(x - a)^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{(x - a)^3}}.$$

$f'(x)$ is infinite for $x = a$, and $f(a) = 0$; but, as x passes through a, $f'(x)$ becomes imaginary.

In this case, it is evident that the value $f(a)$, corresponding to the extreme or limiting value of x, is the least possible value of $f(x)$ as long as we confine ourselves to the positive value of the symbol $(x - a)^{\frac{1}{4}}$; but the curve whose equation is $y^4 = x - a$, which is equivalent to $y = \pm \sqrt[4]{x - a}$, has no minimum ordinate.

109. Again, let us take

$$f(x) = \frac{b^2 x}{(x - a)^2},$$

whence

$$f'(x) = \frac{-b^2(x + a)}{(x - a)^3}.$$

When x passes through the value $-a$, $f'(x)$ changes sign in the order $- 0 +$; hence, $f(-a)$, or $-\frac{b^2}{4a}$, is a minimum. Again, when x passes through the value a, $f'(x)$ passes through infinity and changes sign; but $f(a)$ is also infinite.

MAXIMA AND MINIMA OF A FUNCTION OF A FUNCTION.

110. When it is required to determine the maximum and minimum values of a function f of a variable z, which is not independent, but which is itself a given function φ of another variable x, we may proceed to eliminate z by means of the two equations

$$y = f(z) \quad . \quad . \quad . \quad . \quad (1)$$

and

$$z = \varphi(x), \quad . \quad . \quad . \quad . \quad (2)$$

and then solve as in the preceding sections.

In certain cases, however, the following principle will be found very useful:

Differentiating (1), we obtain

$$dy = f'(z)\,dz. \quad . \quad . \quad . \quad . \quad (3)$$

A maximum or a minimum value of y is attained whenever dy changes sign. Were z an independent variable, it would only be necessary to examine the cases in which $f'(z)$ changed sign, since dz would be arbitrary. In fact, when the independent variable "*passes through* the value a" (Art. 100), its differential cannot change sign.

In the present case, however, z is a function of x; therefore, if z, or $\varphi(x)$, has a maximum or a minimum value, dz will change sign as x passes through a certain value, thus [see Equation (3)] causing a change of sign in dy.* *Hence, we conclude that any function of a maximum or a minimum is a maximum or a minimum.*

111. The form of Equation (3) shows that if $f'(z)$ is positive, the change of sign in dy takes place in the same order as in dz; but if $f'(z)$ is negative, the order is reversed. In other words, if $f(z)$ is an increasing function of z, as z arrives at a maximum or a minimum value, $f(z)$ will simultaneously reach a maximum or minimum value; but if $f(z)$ is a decreasing function, it will reach a minimum when z reaches a maximum value, and *vice versa.*

112. Other solutions may be obtained from Equation (3) whenever the factor $f'(z)$ changes sign. Let a be a value of z thus obtained; it is now necessary to solve the equation

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* As z does not "*pass through*" a maximum or minimum value, neither does $f'(z)$ pass through the value assumed at the instant when this change of sign in dz takes place; it is, therefore, impossible for $f'(z)$ to change sign simultaneously with dz, so as to neutralize the effect of the change of sign of the latter.

$$a = \varphi(x),$$

in order to ascertain whether a is a possible value of z, and, if so, to what values of x it corresponds.

113. Let us take the function $\cos(x^2 - 4x)$. In this case

$$z = x^2 - 4x, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$y = \cos z. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We find from Equation (1) that $x = 2$ gives a minimum value of z equal to -4; since this value of z is between $-\pi$ and $-\frac{3}{2}\pi$, the arc terminates in the second quadrant, where the cosine is a *decreasing* function; and therefore $\cos(-4)$ corresponding to a *minimum* value of z is a *maximum* value of y.

Other solutions may be derived from Equation (2), since we know that $\cos z$ is a maximum when $z = 0, \pm 2\pi$, &c., and a minimum when $z = \pm \pi, \pm 3\pi$, &c.

Putting

$$z = 0 = x^2 - 4x,$$

we derive

$$x = 0 \quad \text{and} \quad x = 4,$$

which give maximum values of y equal to unity.

The general solution of Equation (1) is

$$x = 2 \pm \sqrt{4 + z}.$$

Putting $z = \pi$, we obtain two values of x, giving minimum values of y equal to -1.

Putting $z = -\pi$, we obtain two values of x, giving likewise minimum values of y equal to -1.

In like manner, $z = 2\pi, 3\pi, 4\pi$, &c., give alternate maxima and minima. The remaining negative values of z give imaginary values of x.

THE EMPLOYMENT OF A SUBSTITUTED FUNCTION.

114. In accordance with the principles of the foregoing articles, it is obvious that if $f(z)$ is *always an increasing function*, it will have maxima and minima corresponding to those of z, and *no others.* Such a function may, therefore, be substituted for the given variable z for the purpose of determining its maxima and minima. This substitution will be found advantageous whenever the derivative of the resulting function is more simple than that of the given function.

It is often convenient to apply this method to radical functions, making use of some power of the given function which does not involve radicals. If the exponent indicating the power be a positive odd number, or an *even* number, provided the given function is positive, the resulting expression will be a function that always increases with the given function.

115. The logarithm which is always an increasing function is also frequently employed in a similar way. The logarithm of a negative quantity is, indeed, imaginary; but, since we employ the logarithmic differential, which is the same for a negative as for the corresponding positive quantity (Art. 49), the method is equivalent to the employment of a *decreasing* function *whenever the given quantity becomes negative.*

The case in which the given quantity becomes zero (the logarithm of zero being infinite), must be separately examined, because the logarithmic differential changes sign when the given quantity *passes through zero* as well as when it has a zero maximum or minimum.

When the reciprocal or any other decreasing function is employed in this way, it must be remembered that a *maximum* of the resulting function corresponds to a *minimum* of the given function, and *vice versa.*

116. As an illustration of the application of the logarithmic function, let us take

$$f(x) = \frac{(x+2)^{\frac{1}{2}}\,(x^2+2x)^{\frac{1}{3}}}{(1-x)^{\frac{1}{3}}}.$$

The derivative of the logarithm is

$$\tfrac{1}{2}\cdot\frac{1}{x+2} + \tfrac{2}{3}\cdot\frac{x+1}{x^2+2x} + \frac{1}{3(1-x)}.$$

Reducing, and putting equal zero,

$$\frac{5x^2-7x-4}{6(x^2+2x)(x-1)} = 0.$$

The roots are

$$x = \tfrac{1}{10}(7 \pm \sqrt{129});$$

it is easily proved that the negative root gives a maximum value of $\log[f(x)]$, and that the positive root gives a minimum value. Since $f(x)$ is found on examination to be negative for both roots, its values corresponding to these roots are respectively a minimum and a maximum.

To find the values of x which make the above derivative infinite, we put

$$x(x+2)(x-1)=0.$$

Referring to the value of $f(x)$, it is found to be infinite for $x=1$. As x passes through the root -2, $f(x)$ becomes imaginary; this case is, therefore, similar to that of Art. 108. As x passes through the value 0, $f(x)$ changes sign; this root, therefore, gives neither a maximum nor a minimum.

EXAMPLES XX.

1. $f(x)=(x^2+3x+2)^{\frac{2}{3}}+x^{\frac{2}{3}}$. Show that $f(x)$ has minimum values when x is -2, -1, and 0.

2. $f(x)=(x^2+2x)^{\frac{2}{3}}-(x+3)^{\frac{4}{3}}$.
Maxima for $x=\frac{1}{4}(-3\pm\sqrt{17})$; minima for $x=0$ and $x=-2$.

3. $f(x)=(x-a)^{\frac{4}{3}}(x-b)^{\frac{2}{3}}+c$.
A maximum for $x=\dfrac{2b+a}{3}$; minima for $x=a$ and $x=b$.

4. $f(x)=(x-a)^{\frac{2}{3}}(x-b)^{\frac{1}{3}}$.
Solutions for $x=a$ and $x=\frac{1}{3}(2b+a)$; if $b>a$, the former gives a maximum, and the latter a minimum.

5. $f(x)=\sec x+\log(\pm\cos x)$.

$$f'(x)=(\sec x-1)\tan x.$$

The only finite solutions occur when $x=0$, or $x=\pi$. When x passes through zero, $\sec x-1$ does not change sign, $f'(x)$ changes sign in the order $-\ 0\ +$, indicating a minimum. When x passes through π, the order is $+\ 0\ -$, indicating a maximum.

6. $f(x)=\sec x+\log\cos^2 x$.
Maxima for $x=0$, and $x=\pi$; minima for $x=\pm\frac{1}{3}\pi$.

7. $f(x)=\epsilon^{\sin x\cos^2 x}$.
Maxima for $\cos x=\pm\sqrt{\frac{2}{3}}$; minimum for $\cos x=0$ (the angles being taken in the first semicircle).

8. $f(x)=(a^{x-1})^{x+2}$
Minimum for $x=-\frac{1}{2}$ (a being positive).

9. $f(x)=\dfrac{x}{ax^2-bx+a}$.

Employing the reciprocal, we obtain $x = \pm 1$; $x = -1$ gives a maximum of the reciprocal, and, consequently, a minimum of $f(x)$. $x = +1$ gives a maximum of $f(x)$.

10. $f(x) = \dfrac{x-1}{x^3 - 3x^2 + 2x + 54}$.

The derivative of the reciprocal is

$$2x - 2 - \frac{54}{(x-1)^2};$$

putting it equal to zero, we obtain

$$x = 4,$$

which gives a maximum of $f(x)$. The derivative is infinite for $x = 1$. As x passes through this value, the reciprocal also becomes infinite, and $f(x)$ zero; but, since it changes sign, there is neither a maximum nor a minimum.

11. $f(x) = \sqrt{b^2 + ax} + \sqrt{b^2 - ax}$.

Squaring, we obtain

$$2b^2 + 2\sqrt{b^4 - a^2x^2},$$

which is evidently a maximum for $x = 0$.

12. To divide a number a into a number of equal parts such that their continued product shall be a maximum.

Let x denote the number of parts, then $\frac{a}{x}$ is one of the parts, and $\left(\frac{a}{x}\right)^x$ is the continued product. Taking the logarithmic derivative, and putting it equal to zero, we deduce

$$\log x = \log a - 1 = \log a - \log \varepsilon = \log \frac{a}{\varepsilon},$$

$$x = \frac{a}{\varepsilon}, \quad \text{and} \quad \left(\frac{a}{x}\right)^x = \varepsilon^{\frac{a}{\varepsilon}},$$

a maximum value of the function employed, since the second derivative is negative for positive values of x.

Strictly speaking, this question requires an integral answer. The greatest result is obtained by putting for x one of the two whole numbers between which the value of $\frac{a}{\varepsilon}$ falls.

13. $f(x) = \dfrac{\sin x \cos x}{\cos^2 (\frac{1}{3}\pi - x)}$.

Taking the logarithmic derivative, we have

$$\cot x - \tan x - 2\tan\left(\tfrac{1}{3}\pi - x\right);$$

substituting the value of $\tan\left(\frac{1}{3}\pi - x\right)$, and putting this derivative equal to zero, we have

$$\cot x - \tan x - \frac{2\sqrt{3} - 2\tan x}{1 + \sqrt{3}\tan x} = 0;$$

reducing, we obtain

$$\sec^2 x\,(1 - \sqrt{3}\tan x) = 0.$$

The only solution is

$$\tan x = \tfrac{1}{3}\sqrt{3}.$$

Whence,

$$x = \tfrac{1}{6}\pi \quad \text{or} \quad \tfrac{7}{6}\pi, \text{ giving maxima.}$$

The derivative is infinite for $x = 0$, $x = \frac{1}{2}\pi$, and $x = -\frac{1}{6}\pi$; the last renders $f(x)$ infinite. When x passes through the first and second values, $f(x)$ changes sign, and therefore in these cases, it is neither a maximum nor a minimum.

The employment of the reciprocal function furnishes a simpler solution.

On expanding $\cos^2\left(\frac{1}{3}\pi - x\right)$, the reciprocal reduces to

$$\tfrac{1}{4}\left(\cot x + 3\tan x + 2\sqrt{3}\right).$$

Putting the derivative equal to zero, we obtain

$$-\operatorname{cosec}^2 x + 3\sec^2 x = 0.$$

Whence,

$$\tan x = \pm\tfrac{1}{3}\sqrt{3},$$

and therefore

$$x = \pm\tfrac{1}{6}\pi \quad \text{and} \quad \pm\tfrac{7}{6}\pi.$$

The positive value of $\tan x$ gives a minimum, and the negative value a maximum value of the reciprocal. This maximum is, however, equal to zero, and corresponds to an infinite value of $f(x)$.

14. The lower corner of a leaf of a book is folded over so as just to reach the inner edge of the page; find when the crease thus formed is a minimum.

Let y denote the length of the crease, x the distance of the corner from the intersection of the crease with the lower edge, and a the width of the page.

By means of the relations of similar right triangles, the following expression is deduced:

$$y = \frac{x\sqrt{x}}{\sqrt{x - \frac{1}{2}a}}.$$

Whence we obtain

$$x = \tfrac{3}{4}\,a,$$

which gives a minimum value of y.

15. Find when the area of the part folded over is a minimum.

When $x = \frac{2}{3}\,a$.

XXI.

MAXIMA AND MINIMA OF IMPLICIT FUNCTIONS.

117. To determine the maximum and minimum values of y when the relation between y and x is such that y cannot conveniently be expressed as an explicit function, we derive the value of $\frac{dy}{dx}$ in terms of x and y by the method explained in Art. 81. The equation thus obtained will be of the form

$$\frac{dy}{dx}=\frac{u}{v}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which u and v denote, in general, functions of x and y. In the case of a maximum or a minimum value of y, we have, as usual, the condition

$$\frac{dy}{dx}=0,$$

whence,

$$u=0, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

an equation involving x and y, which must be combined with the original equation in order to determine x.

To decide whether a value of x thus obtained corresponds to a maximum or a minimum, we examine the second, and, if necessary, higher derivatives. From (1) we obtain

$$\frac{d^2y}{dx^2}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}; \quad . \quad . \quad . \quad . \quad (3)$$

but, since $\frac{dy}{dx}$ (and therefore u) is in this case zero, (3) reduces to

$$\left[\frac{d^2y}{dx^2}\right]=\frac{\left[\frac{du}{dx}\right]}{v}, \quad . \quad . \quad . \quad . \quad (4)$$

in which the notation [ ] denotes the value of the enclosed derivative when $\frac{dy}{dx}=0$.

As an illustration, let us take

$$xy^2-x^2y=2a^3, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence

$$y^2 + 2xy\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0;$$

therefore,

$$\frac{dy}{dx} = \frac{y(2x - y)}{x(2y - x)}. \quad . \quad . \quad . \quad (2)$$

By the condition for a maximum or a minimum, we have

$$u = y(2x - y) = 0;$$

whence

$$y = 0 \quad \text{or} \quad y = 2x.$$

Putting $y = 0$ in (1) gives an impossible value of x, but $y = 2x$ gives

$$x = a \quad \text{and} \quad y = 2a.$$

From (2) we obtain

$$\left[\frac{d^2y}{dx^2}\right] = \frac{y}{x(2y - x)},$$

which is positive for $x = a$; the corresponding value of y is, therefore, a *minimum*.

In this example, y has two values for each value of x, since the implicit function includes two explicit functions. We have, in fact, found the minimum ordinate of the curve represented by the given equation.

The results derived by making $\frac{dy}{dx}$ infinite correspond to points at which the tangent to the curve is parallel to the axis of y. The two values of y here become equal, hence the two branches of the curve meet. At such a point, the curve has a limiting abscissa [compare Art. 108]. In fact, the condition $\frac{dy}{dx} = \infty$ is equivalent to $\frac{dx}{dy} = 0$, which would be used in determining maxima and minima abscissas.

A CLASS OF PROBLEMS INVOLVING MORE THAN ONE INDEPENDENT VARIABLE.

118. The determination of the maxima and minima of functions of several independent variables depends, in general, upon principles established in the chapter containing the complete discussion of such functions; but certain cases of frequent occurrence may be solved in the manner illustrated in the following example:

Required, the dimensions of an open rectangular box of given capacity requiring for its construction the least possible amount of material.

In this example, there are three variables (the three dimensions), connected only by the condition that their product shall be a given constant; hence, there are in effect two independent variables, x and y. Denoting the height by z and the given capacity by c^3, we have

$$xyz = c^3. \quad . \quad . \quad . \quad . \quad (1)$$

The expression for the area is

$$xy + 2(x + y)z. . \quad . \quad . \quad . \quad (2)$$

We first show that when this expression has a minimum value, we must have $x = y$. Let B denote the unknown value of the area of the base xy when the expression is a minimum; then, eliminating z by Equation (1), and substituting B for xy, (2) becomes

$$B + 2(x + y)\frac{c^3}{B}.$$

Now, in order that the entire expression may have the minimum value required, we must select from the values of x and y which satisfy the equation

$$xy = B$$

such values as will make the factor $(x + y)$ a minimum; this will be the case when $x = y$ (Example 13, XVI). The base of the box is, therefore, a square, and (2) reduces to

$$x^2 + \frac{4c^3}{x}.$$

Whence, differentiating,

$$2x - \frac{4c^3}{x^2} = 0;$$

therefore,

$$x = c\sqrt[3]{2}.$$

and

$$z = \tfrac{1}{2}c\sqrt[3]{2}.$$

119. When the shape of the box only is required, we may proceed thus:—Assuming the base to be a square, we have

$$x^2 + 4xz = \text{a minimum}, \quad . \quad . \quad . \quad (3)$$

and

$$x^2 z = c^3; \quad . \quad . \quad . \quad . \quad (4)$$

whence,

$$x\,dx + 2x\,dz + 2z\,dx = 0,$$

and

$$2xz\,dx + x^2\,dz = 0.$$

Eliminating $\frac{dx}{dz}$, we obtain

$$x = 2z.$$

The height of the box is, therefore, half the edge of the base.

CORRELATED PROBLEMS OF MAXIMA AND MINIMA.

120. Many problems of maxima and minima have a reciprocal relation, such that the solution of one problem being known, that of the other may at once be inferred. As an illustration, we introduce the reciprocal problem to the one solved in the preceding article.

An open rectangular box is to be constructed from a given amount of material; required, its shape in order that it shall have the greatest possible capacity.

We shall prove that the shape of the box in this case is the same as that found in the last problem for the box requiring a minimum of material for a given capacity.

Let the minimum of material corresponding to a given capacity c^3 be denoted by k^2; k'^2 denoting in like manner the minimum corresponding to c'^3.

It is evident, since the shape remains the same, that k^2 and c^3 increase and decrease together.

Now, let k^2 denote the *given* amount of material; then, *if the box of maximum capacity has not the shape specified*, its capacity will be different from c^3; denote it by c'^3. Now, without changing the capacity, we may *decrease* the amount of material to the corresponding minimum k'^2. Then, since

$$k'^2 < k^2, \quad c'^3 < c^3,$$

which is contrary to the hypothesis that c'^3 is the maximum capacity. Therefore, *the box of maximum capacity must have* *the same shape as the one requiring a minimum of material with a given capacity; that is, its height is one-half the edge of the base.*

121. The analytical reason for this correspondence of solutions is evident on referring to Equations (3) and (4) of Art. 119. In the problem just discussed, we should have, instead of (3) and (4), the equations

$$x^2 z = \text{a maximum},$$

and

$$x^2 + 4xz = k^2,$$

which would give the same differential equations for determining the shape.

If the quantities which take the place of c^3 and k^2 are such that, instead of increasing together as in the above case, the one increases as the other diminishes, it may be shown that a maximum solution of one problem will correspond to a maximum solution of the other, and a minimum to a minimum

FERMAT'S PROBLEM.

122. *The position of a point in each of two media (A and B) separated by a plane surface, being given, it is required to find the path described by a particle which goes from one point to the other in the shortest possible time.* The velocity of the particle, which is constant in each medium, is denoted, in the medium A by u, and in the medium B by v.

The path is evidently composed of two straight lines, one in each medium. It is easily proved that the plane passing through the two points, and normal to the surface, contains these lines.

For, if not, let the path be projected on this plane, then the portion of the new path thus formed, contained in each medium, will be shorter than the corresponding portion of the original path; therefore the new path will be described in less time than the original path.

123. Let a and b denote the lengths of the perpendiculars to the separating surface let fall from the given points, and c the projection of the path on this surface.

Let the particle be supposed to move from A to B, and let i denote the angle of incidence, that is, the angle between a

and the path in A, and r the corresponding angle in the medium B.

The time occupied in describing the portion of the path in the medium A is expressed by $\frac{a \sec i}{u}$, and the time in the medium B by $\frac{b \sec r}{v}$; hence the function to be made a minimum is

$$\frac{a \sec i}{u} + \frac{b \sec r}{v}, \quad . \quad . \quad . \quad (1)$$

the relation between i and r being expressed by

$$a \tan i + b \tan r = c. \quad . \quad . \quad . \quad (2)$$

The differential of (1) is

$$\frac{a}{u} \sec i \tan i \, di + \frac{b}{v} \sec r \tan r \, dr;$$

putting this equal to zero, we obtain

$$\frac{a}{u} \sec i \tan i \, di = -\frac{b}{v} \sec r \tan r \, dr. \quad . \quad . \quad (3)$$

From (2) we obtain

$$a \sec^2 i \, di = -b \sec^2 r \, dr; \quad . \quad . \quad (4)$$

dividing (3) by (4) to eliminate di and dr, we have

$$\frac{1}{u} \cdot \frac{\tan i}{\sec i} = \frac{1}{v} \cdot \frac{\tan r}{\sec r},$$

or

$$\frac{v}{u} = \frac{\sin r}{\sin i}.$$

124. This result has been employed to show that the hypothesis that the path of a ray of light is that which occupies the least time is consistent with the observed fact, that the ratio of the sine of the angle of incidence to the sine of the angle of refraction is constant, provided we suppose the velocities to increase with the densities. This last supposition is consistent with the undulatory theory, but inconsistent with the transmission theory.

EXAMPLES XXI.

MISCELLANEOUS.

1. $5y^2 - 6xy + x^2 - 9 = 0$: find maxima and minima of y.

The positive value of x corresponds to a maximum, and the negative to a minimum.

2. $y^2 - x^2y + x - x^3 = 0$. Prove that $x = -1$ gives a maximum value of y.

3. $x^4 + 2ax^2y - ay^3 = 0$. Show that when $x = \pm a$, $y = -a$ and is a minimum.

4. $3a^2y^2 + xy^3 + 4ax^3 = 0$. Show that when $x = \frac{3}{2}a$, y has a maximum value, namely $-3a$, the value of $\left[\frac{d^2y}{dx^2}\right]$ being then $-\frac{8}{5a}$.

5. In a given cone inscribe the greatest rectangular parallelopiped. [See Art. 118.]

The base is a square, and the height is equal to one-third the altitude.

6. In a given sphere inscribe the greatest rectangular parallelopiped. [See Art. 118.]

A cube.

7. $f(x) = \frac{x}{1 + x\tan x}$.

A maximum when $x = \cos x$.

8. $f(x) = \frac{\tan^3 x}{\tan 3x}$.

A minimum for $x = 0$, $\frac{3}{8}\pi$, $\frac{5}{8}\pi$, and π: a maximum for $x = \frac{1}{8}\pi$, $\frac{1}{2}\pi$, $\frac{7}{8}\pi$, &c.

9. $f(x) = \frac{x^2 + 2x + 11}{x^2 + 4x + 10}$.

10. $f(x) = \frac{\log x}{x^n}$. A maximum for $x = \varepsilon^{\frac{1}{n}}$.

11. $f(x) = (x - 1)^3 (x + 1)^{-2}$.

12. $f(x) = (1 + x^{\frac{1}{3}}) (7 - x)^2$.

Solve by putting $x = z^3$. A minimum for $x = 0$ and $x = 7$: a maximum for $x = 1$.

13. $f(x) = 5x^6 + 12x^5 - 15x^4 - 40x^3 + 15x^2 + 60x + 27$.

$x = -2$ gives a minimum; $x = +1$ gives neither a maximum nor a minimum.

14. $f(x) = \dfrac{(a-x)^3}{a-2x}$.

$x = \frac{1}{4}a$ gives a minimum.

15. Determine the cylinder inscribed in a sphere so that its whole surface shall be a maximum.

$$\text{Height} = r\left[2\left(1 - \tfrac{1}{5}\sqrt{5}\right)\right]^{\frac{1}{2}}.$$

16. Determine the cone inscribed in a given sphere so that its whole surface shall be a maximum.

$$\text{Height} = \frac{r}{16}(23 - \sqrt{17}).$$

17. Given the vertical angle of a triangle and its area; find when its base is a minimum.

8. It is required to construct from two circular iron plates of radius a a buoy composed of two equal cones having a common base which shall have the greatest possible volume.

19. Find the cone of minimum volume described about a given sphere.

The height is twice the diameter of the sphere.

20. A cylindrical trough is constructed by bending a given sheet of tin; its breadth being denoted by $2a$, find the radius of the cylinder when the capacity of the trough is a maximum.

Let x denote the radius of the cylinder; then $\dfrac{a}{x}$ will denote the half-angle of the segment which forms the trough. The area of any section will be expressed by

$$ax - x\cos\frac{a}{x}\,.\,x\sin\frac{a}{x};$$

therefore

$$f(x) = ax - x^2\sin\frac{a}{x}\cos\frac{a}{x},$$

and

$$f'(x) = 2a\cos^2\frac{a}{x} - 2x\sin\frac{a}{x}\cos\frac{a}{x}.$$

Whence

$$\cos\frac{a}{x}\left(a\cos\frac{a}{x} - x\sin\frac{a}{x}\right) = 0,$$

$$\cos\frac{a}{x} = 0 \quad \therefore \quad \frac{2a}{x} = \pi.$$

There is evidently a maximum between $\frac{a}{x} = 0$ and $\frac{a}{x} = \pi$; and as $\frac{a}{x} = \frac{\pi}{2}$ is the only root between these limits, it must correspond to the maximum sought.

21. A steamer whose speed is 8 knots per hour and course due north sights another steamer directly ahead, whose speed is 10 knots, and whose course is due west. What must be the course of the first steamer to cross the track of the second at the least possible distance from her?

Result. N. 53° 8′ *W.*

22. Determine the angle which a rudder makes with the keel of a ship when its turning effect is the greatest possible.

Let φ denote the angle between the rudder and the prolongation of the keel of the ship; then the stream of water intercepted will be proportional to $\sin\varphi$: the resulting force being decomposed, the component perpendicular to the rudder will be proportional to $\sin^2\varphi$. Again decomposing this force, and taking the component that is perpendicular to the keel of the ship, which is the only part of the original force that is effective in turning the ship, we have the expression

$$\sin^2\varphi\cos\varphi.$$

Whence we obtain

$$\sin\varphi = \tfrac{1}{3}\sqrt{6}.$$

23. The work of driving a steamer through the water being proportional to the cube of her speed, find her most economical rate per hour against a current running a knots per hour.

Let v denote the speed of the steamer in knots per hour. The work per hour will then be denoted by kv^3, k being some constant, and the actual distance the steamer advances per hour by $v - a$. The work per knot is expressed by

$$\frac{kv^3}{v-a}.$$

Whence we obtain the result

$$v = \tfrac{3}{2}a.$$

24. *To find the parallel on the earth's surface at which the dif-*

ference between the geographical and the geocentric latitude is the greatest.

Let us take any meridian; assuming it to be an ellipse, and taking the origin at the centre, its equation will be

$$y = \frac{b}{a}\sqrt{a^2 - x^2}, \text{ for the upper branch.}$$

Let θ denote the geocentric latitude, and ψ the geographical latitude;

$$\tan\theta = \frac{y}{x} = \frac{b\sqrt{a^2 - x^2}}{ax},$$

and, since $\psi = \varphi - 90$,

$$\tan\psi = -\cot\varphi = -\frac{dx}{dy} = \frac{a\sqrt{a^2 - x^2}}{bx}.$$

Since the tangent is an increasing function, we may use $\tan(\psi - \theta)$ instead of $\psi - \theta$.

$$\tan(\psi - \theta) = \frac{\tan\psi - \tan\theta}{1 + \tan\psi\tan\theta};$$

substituting, we have

$$\tan(\psi - \theta) = \frac{\dfrac{a\sqrt{a^2 - x^2}}{bx} - \dfrac{b\sqrt{a^2 - x^2}}{ax}}{1 + \dfrac{a^2 - x^2}{x^2}};$$

$$\tan(\psi - \theta) = \frac{x\sqrt{a^2 - x^2}}{ab} - \frac{bx\sqrt{a^2 - x^2}}{a^3},$$

$$\tan(\psi - \theta) = \left(\frac{1}{ab} - \frac{b}{a^3}\right)x\sqrt{a^2 - x^2}.$$

Squaring and rejecting the constant, we put

$$f(x) = a^2x^2 - x^4,$$

then

$$f'(x) = 2a^2x - 4x^3.$$

Whence

$$x = 0, \quad \text{or} \quad x = \pm\frac{a}{\sqrt{2}}.$$

The root $x = \frac{a}{\sqrt{2}}$ corresponds to the maximum required. The function $x\sqrt{a^2 - x^2}$ passes through zero, and changes sign at $x = 0$.

CHAPTER VI.

THE DEVELOPMENT OF FUNCTIONS IN SERIES.

XXII.

THE NATURE OF AN INFINITE SERIES.

125. A function which can be expressed by means of a limited number of integral terms, involving powers of the independent variable with positive integral exponents only, is called a *rational integral function.*

When $f(x)$ is *not* a rational integral function, there usually exists an unlimited series of terms rational and integral in x, which series is regarded as an algebraic equivalent for the function. The process of deriving this series is called the *development* of the function into an *infinite series.*

126. A familiar mode of development is that of division, the function being a rational fraction. Thus,

$$\frac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + \&c.,$$

a series of terms arranged according to ascending powers of x. In this case, the law of the series is apparent; each coefficient after the absolute term being 2.

It is to be observed, in the first place, that, owing to the indefinite number of terms in the second member, this equation cannot be verified numerically for an assumed value of x. In order that this may be done, the remainder after any number $(n+1)$ of terms of the series must be written; thus,

$$\frac{1+x}{1-x} = 1 + 2x + 2x^2 \;.\;.\;.\; + 2x^n + \frac{2x^{n+1}}{1-x}.$$

This equation may now be verified numerically for any assumed value of x; or algebraically by multiplying each member by $1 - x$, thus obtaining an identity.

127. The ordinary process of extracting the square root of a polynomial furnishes an example of a series which may be extended so as to include as many terms as we please; but this process gives us no expression for the remainder. When this expression is not given, no complete verification of the equation between the function and the series is possible. Nevertheless (the remainder being denoted by R), the correctness of the coefficients up to a given point in the series may be shown by a process which does not lead to an identity, but which leaves an equation containing a limited number of terms, and involving the unknown remainder R. We assume then that R is such a function as will satisfy the equation which remains after the coefficients are thus determined.

128. If it is assumed that a function admits of development into a series involving ascending powers of x, we may write

$$f(x) = A + Bx + Cx^2 \dots + Nx^n + R, \quad . \quad (1)$$

$A, B, C, \dots N$ being coefficients independent of x, and as yet unknown; R, the remainder, is however a function of x and also of n.

Thus, we may assume

$$\frac{1}{1-x} = A + Bx + Cx^2 \dots + Nx^n + R.$$

If we multiply by $(1-x)$ (which is the process we should employ to verify the series were the coefficients known), we have

$$1 = A \begin{array}{l|} +B \\ -A \end{array} \, x \begin{array}{l|} +C \\ -B \end{array} \, x^2 \dots \begin{array}{l|} +N \\ -M \end{array} \, x^n - Nx^{n+1} + R(1-x),$$

which must reduce to an identity.

The unknown coefficients of this equation of verification may now be so determined that, however great the value of n, all the terms except the last two shall vanish. Thus, assuming the absolute term, and also the terms involving each power of x separately to vanish, we obtain

$$A = 1, \; B = A = 1, \; C = B = 1, \dots N = 1.$$

These equations determine the law of the succession of the coefficients, and reduce the equation of verification to

$$0 = -x^{n+1} + R(1-x).$$

Whence we derive

$$R = \frac{x^{n+1}}{1-x},$$

which determines the value of R.

129. The assumption that the absolute terms vanish is equivalent to assuming that the remainder vanishes with x, since all the remaining terms of the equation of verification, except the one containing R, necessarily vanish for $x = 0$. Putting $x = 0$ in Equation (1), we have therefore

$$f(0) = A, \quad . \quad . \quad . \quad . \quad (2)$$

that is, the absolute term of the development is the value the function assumes when $x = 0$.

130. If, in any case, the assumed development be impossible, the fact will be made apparent by the occurrence of impossible conditions requiring infinite values for some of the coefficients. Thus, if $\log x$ be the function of x in Equation (1), it is evident, by Equation (2), that A becomes infinite; the development of $\log x$ in the assumed form is therefore impossible.

CONVERGENT AND DIVERGENT SERIES.

131. The substitution of a particular value for x in an algebraic series gives a series of numerical terms. If, for this particular value of x, the remainder decreases as n increases, it is plain that the more terms we take the nearer will their sum approach the value of the equivalent function, and that the values of the successive terms will in general decrease.

If at the same time the value of the function be finite, the series is said to be *convergent*, since the sum of n terms approximates more and more closely to the value of the function as n increases.

132. It must not however be assumed that, whenever the successive terms decrease for a particular value of x, the series is convergent; for it may happen that, for this value of x, $f(x)$ is infinite, in which case the remainder will also be infinite, whatever be the value of n. Thus the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \ . \ . \ . \ \&c.$$

is not called a convergent series, although the successive terms diminish, because, as a matter of fact, the sum of n terms of this series increases without limit as n increases. This series is, in fact, obtained by giving x the value unity in the development of $\log \dfrac{1}{1-x}$ or $-\log(1-x)$, the function being infinite for this value of x. [See Eq. (2), Art. 145.]

133. When the successive terms of a series increase while the value of the function remains finite, it is evident that the remainder must *increase* as n increases. In this case the series is said to be *divergent.*

Every algebraic series is evidently convergent for values of x below a certain limit, since the remainder vanishes with x.

It is obvious that an algebraic series can be used for computing the value of the equivalent function only when the value of x is such as to render it convergent.

TAYLOR'S THEOREM.

134. The object of Taylor's Theorem is to develop a function into a series involving ascending powers of some quantity upon which it depends. We shall denote the function by $f(x_0+h)$, h being the quantity whose powers appear in the development.

Assume then

$$f(x_0+h) = A_0 + B_0h + C_0h^2 + \dots + N_0h^n + R_0, \quad (1)$$

in which A_0, B_0, C_0, ... N_0 depend upon x_0, but are independent of h, while R_0 is dependent both upon x_0 and upon h, and is such a function of the latter as to vanish when h equals zero. Hence, making $h=0$, we have

$$f(x_0) = A_0.$$

Let $x_1 = x_0 + h$, then

$$h = x_1 - x_0,$$

substituting, (1) takes the form

$$f(x_1) = f(x_0) + B_0(x_1-x_0) + C_0(x_1-x_0)^2 \dots + N_0(x_1-x_0)^n + R_0.$$

We shall now regard x_0 as variable, and denote its general value by x; and, since B_0, ... N_0, and R_0 are functions of x_0, we replace them also by general values B, ... N, and R, R being a function not only of x but also of x_1 and of n. Thus we have

$$f(x_1) = f(x) + B(x_1-x) + C(x_1-x)^2 \dots + N(x_1+x)^n + R. \quad (2)$$

This equation being assumed to be identical, its derivative taken with reference to x will also be identical. Whence

$$0 = f'(x) - B + (x_1-x)\frac{dB}{dx} - 2C(x_1-x) + (x_1-x)^2\frac{dC}{dx} \dots\dots$$
$$- nN(x_1-x)^{n-1} + (x_1-x)^n\frac{dN}{dx} + \frac{dR}{dx}. \quad (3)$$

In order that this equation may be satisfied independently of the value of x, and however great the value of n, it is necessary that the coefficients of the different powers of $(x_1 - x)$ shall separately vanish; hence

$$B = f'(x), \quad \text{and} \quad C = \tfrac{1}{2}\frac{dB}{dx},$$

or, substituting for B its value $f'(x)$,

$$C = \tfrac{1}{2}f''(x).$$

In like manner,

$$D = \tfrac{1}{3}\cdot\frac{dC}{dx} = \frac{1}{1.2.3}f'''(x), \text{ \&c.}$$

Finally,

$$N = \frac{1}{1.2\ldots n}f^n(x).$$

Putting x_0 for x, and substituting in Equation (1) the values of $A_0, B_0, C_0, \ldots N_0$, we obtain

$$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{1.2}\ldots f^n(x_0)\frac{h^n}{1.2\ldots n} + R_0.$$

The law of succession of terms expressed in this equation is called Taylor's Theorem, from the name of its discoverer, Dr. Brook Taylor, who first published it in 1715.

EXPRESSIONS FOR THE REMAINDER.

135. We shall now proceed to derive an expression for the remainder R_0, and at the same time to determine when the above series may be employed in computing the values of functions.

To derive an equation for determining R, we substitute in Equation (3) the values of $B, C, \ldots N$, already determined, and obtain

$$0 = \frac{1}{1.2.3\ldots n}f^{n+1}(x)(x_1 - x)^n + \frac{dR}{dx}. \quad . \quad (4)$$

It is obvious from the values of $B, C, \ldots N$, and from Eq. (2) that, if $f(x)$ *and all its derivatives up to the nth inclusive vary continuously as* x *varies from* x_0 *to* x_1, R will also vary continuously from R_0 to 0 (the latter being the value of R which corresponds to $x = x_1$).

136. Now let P denote an assumed function of x, which likewise varies continuously from R_0 to 0 as x varies from x_0

to x_1. Now, since R and P both vary continuously, starting together at the value R_0, and reaching the value zero together, it is impossible that one of them should vary throughout this interval with a more rapid rate than the other. Hence, *there must be at least one intermediate value of x for which the derivatives of R and P are equal.* Let such an intermediate value be denoted by $x_0 + \theta h$, θ being a proper fraction.

From (4) we obtain

$$\frac{dR}{dx} = -\frac{1}{1.2\ldots n} f^{n+1}(x)\,(x_1 - x)^n;$$

putting $x_0 + \theta h$ for x, whence

$$x_1 - x = x_1 - x_0 - \theta h = h\,(1 - \theta),$$

we have

$$\left.\frac{dR}{dx}\right]^{*}_{x_0 + \theta h} = -\frac{h^n}{1.2\ldots n} f^{n+1}(x_0 + \theta h)\,(1 - \theta)^n. \quad . \quad (5)$$

LAGRANGE'S FORM OF THE REMAINDER.

137. If we assume the function P in the form

$$P = R_0 \frac{(x_1 - x)^{n+1}}{h^{n+1}},$$

the conditions required in defining P (Art. 136) will be fulfilled. Taking the derivative

$$\frac{dP}{dx} = -\frac{R_0\,(n+1)\,(x_1 - x)^n}{h^{n+1}},$$

and

$$\left.\frac{dP}{dx}\right]_{x_0 + \theta h} = -\frac{n+1}{h} R_0\,(1 - \theta)^n. \quad . \quad . \quad (6)$$

Equating the values of the derivatives in Equations (5) and (6), we have

$$R_0 = f^{n+1}(x_0 + \theta h)\,\frac{h^{n+1}}{1.2\ldots\ldots(n+1)}.$$

This form of the remainder is due to Lagrange. We may now write the series thus:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{1.2}\ldots$$
$$+ f^n(x_0)\frac{h^n}{1.2\ldots n} + f^{n+1}(x_0 + \theta h)\frac{h^{n+1}}{1.2\ldots(n+1)}. \quad (7)$$

---

* The symbol $]_a$ placed after a derivative indicates the value of the derivative when the quantity denoted by the subscript is substituted for the variable x.

ANOTHER FORM OF THE REMAINDER.

138. The function P will also fulfil the requisite conditions when assumed in the form*

$$P = R_0 \frac{f^n(x_1) - f^n(x)}{f^n(x_1) - f^n(x_0)};$$

since $f^n(x)$ is, by the hypothesis of Art. 135, continuous between the limits $x = x_0$ and $x = x_1$; and since the function is equal to R_0 when $x = x_0$, and is zero when $x = x_1$.

Taking the derivative

$$\frac{dP}{dx} = - R_0 \frac{f^{n+1}(x)}{f^n(x_1) - f^n(x_0)},$$

and

$$\left.\frac{dP}{dx}\right]_{x_0 + \theta' h} = - R_0 \frac{f^{n+1}(x_0 + \theta' h)}{f^n(x_1) - f^n(x_0)}.$$

Equating this expression with the value of the derivative in Equation (5), and reducing, we obtain

$$R_0 = (1 - \theta')^n \left[f^n(x_1) - f^n(x_0) \right] \frac{h^n}{1 . 2 \, . . \, n}.$$

Writing θ in place of $(1 - \theta')^n$, since the latter is a proper fraction, substituting this expression for R_0 in the series, and arranging terms, we obtain

$$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0) \frac{h^2}{1 . 2} \cdots$$

$$+ f^{n-1}(x_0) \frac{h^{n-1}}{1 . 2 \, . . \, (n-1)} + \left[(1-\theta) f^n(x_0) + \theta f^n(x_0 + h) \right] \frac{h^n}{1 . 2 \, . . \, n}. \quad (8)$$

139. The coefficient of $\frac{h^n}{1 . 2 \ldots n}$ in this form of the series has evidently a value between $f^n(x_0)$ and $f^n(x_1)$.

It must be remembered that the completed forms of Taylor's Theorem, expressed in Equations (7) and (8), are demonstrated only on the hypothesis of Art. 135. The expressions

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* The form

$$P = R_0 \frac{\phi(x_1) - \phi(x)}{\phi(x_1) - \phi(x_0)},$$

in which ϕ is continuous between $\phi(x_1)$ and $\phi(x_0)$, includes all the forms in which P can be assumed. The resulting value of R_0 is

$$R_0 = \theta [\phi(x_1) - \phi(x_0)] \frac{f^{n+1}(x_0 + \theta h)}{\phi'(x_0 + \theta h)} \cdot \frac{h^n}{1 . 2 \ldots n},$$

which is accordingly the most general form of R_0 that can be derived by this method. See *The Mathematical Messenger* for April, 1873, in which this method of treating Taylor's Theorum was first published.

for the remainder are indefinite, inasmuch as they contain the unknown quantity θ; but, since θ is less than unity, they are frequently useful in determining the limit to the error committed when we stop at a given term of a converging series.

EXAMPLES XXII.

1. To expand $\log(x_0 + h)$ by Taylor's Theorem.

In this case

$$f(x_0 + h) = \log(x_0 + h)$$

$$f(x) = \log x \quad \therefore \quad f(x_0) = \log x_0$$

$$f'(x) = \frac{1}{x} \quad \therefore \quad f'(x_0) = \frac{1}{x_0}$$

$$f''(x) = -\frac{1}{x^2} \quad \therefore \quad f''(x_0) = -\frac{1}{x_0^2}$$

$$f'''(x) = \frac{2}{x^3} \quad \therefore \quad f'''(x_0) = \frac{1 . 2}{x_0^3}$$

$$f^{\text{iv}}(x) = -\frac{2 . 3}{x^4} \quad \therefore \quad f^{\text{iv}}(x_0) = -\frac{1 . 2 . 3}{x_0^4}$$

. . . .

$$f^n(x) = -(-1)^n \frac{1 . 2 \ldots (n-1)}{x^n} \quad \therefore f^n(x_0) = -(-1)^n \frac{1 . 2 \ldots (n-1)}{x_0^n}.$$

The second form of the remainder being, in this case, somewhat more convenient than the first, we substitute in Equation (8), and thus obtain

$$\log(x_0 + h) = \log x_0 + \frac{h}{x_0} - \frac{h^2}{2x_0^2} + \frac{h^3}{3x_0^3} - \frac{h^4}{4x_0^4} \ldots .$$
$$- (-1)^n \left[\frac{1 - \theta}{x_0^n} + \frac{\theta}{(x_0 + h)^n} \right] \frac{h^n}{n}.$$

2. Expand $a^{x_0 + h}$.

In this case,

$$f(x_0 + h) = a^{x_0 + h}$$

$$f(x) = a^x \quad \therefore \quad f(x_0) = a^{x_0}$$

$$f'(x) = \log a . a^x \quad \therefore \quad f'(x_0) = \log a . a^{x_0}$$

$$f''(x) = (\log a)^2 . a^x \quad \therefore \quad f''(x_0) = (\log a)^2 . a^{x_0}$$

. . . .

$$f^n(x) = (\log a)^n . a^x \quad \therefore \quad f^n(x_0) = (\log a)^n . a^x$$

Substituting in Equation (7), we have

$$a^{x_0 + h} = a^{x_0} + \log a . a^{x_0} h + (\log a)^2 . a^{x_0} \frac{h^2}{1 . 2} + \ldots ..$$
$$\frac{(\log a)^n . a^{x_0} h^n}{1 . 2 \ldots n} + \frac{(\log a)^{n+1} a^{x_0 + \theta h} h^{n+1}}{1 . 2 \ldots (n+1)}.$$

3. Expand $(a+b)^m$ into a series of terms involving ascending powers of b.

In this example, b takes the place of h and a that of x_0; hence expand $(x_0+h)^m$ by Taylor's Theorem thus:

$$\begin{aligned} f(x)&=x^m & \therefore\ f(x_0)&=x_0^m &&=a^m\\ f'(x)&=mx^{m-1} & \therefore\ f'(x_0)&=mx_0^{m-1} &&=ma^{m-1}\\ f''(x)&=m(m-1)x^{m-2} & \therefore f''(x_0)&=m(m-1)x_0^{m-2}&&=m(m-1)a^{m-2}\end{aligned}$$

Whence

$$(a+b)^m=a^m+ma^{m-1}b+\frac{m(m-1)}{1.2}a^{m-2}b^2\ \ldots.$$
$$+\frac{m(m-1)(m-2)\ldots(m-n+1)}{1.2.3\ldots n}a^{m-n}b^n+\&c.$$

The expression for the remainder, obtained by the formula of Art. 137, is

$$\frac{m(m-1)(m-2)\ldots(m-n)}{1.2.3\ldots\ldots(n+1)}(a+\theta b)^{m-n-1}b^{n+1}.$$

When m is a positive integer, the remainder becomes zero for $n=m$, it is therefore obvious that, in this case, the series contains a finite number of terms.

The above development of $(a+b)^m$ is called the *Binomial Theorem.*

4. Expand $\sin^{-1}(x_0+h)$ to the fourth term inclusive.

$$\sin^{-1}(x_0+h)=\sin^{-1}x_0+\frac{h}{(1-x_0^2)}+\frac{x_0}{(1-x_0^2)^{\frac{3}{2}}}\cdot\frac{h^2}{1.2}\ \ldots$$
$$+\frac{1+2x_0^2}{(1-x_0^2)^{\frac{5}{2}}}\cdot\frac{h^3}{1.2.3}+\&c.$$

When, as in this example, the general expressions for $f^n(x_0)$ and for the remainder cannot easily be obtained, it is nevertheless possible to employ a few terms of the series when it is sufficiently convergent.

5. Prove that

$$\sin(\tfrac{1}{6}\pi+h)=\tfrac{1}{2}\Big(1+h\sqrt{3}-\frac{h^2}{1.2}-\frac{h^3\sqrt{3}}{1.2.3}+\frac{h^4}{1.2.3.4}$$
$$+\frac{h^5\sqrt{3}}{1.2.3.4.5}-\&c.\Big)$$

6. Prove that

$$\tan(\tfrac{1}{4}\pi+h)=1+2h+2h^2+\tfrac{8}{3}h^3+\tfrac{10}{3}h^4+\&c.$$

XXIII.

MACLAURIN'S THEOREM.

140. We shall now give a particular form of Taylor's Series, which is usually more convenient when numerical results are to be obtained than the general forms given in XXII.

This form of the series, although first discovered by Sterling, has received the name of Maclaurin's Series. It was published by Maclaurin in 1742; but it does not appear that he ever intended to claim it as his own discovery.

The form in question, and corresponding expressions for the remainder, are obtained by putting $x_0 = 0$, and replacing h by x in Equation (7) or (8); thus:

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{1 \, . \, 2} \dots \\ + f^n(0)\frac{x^n}{1 \, . \, 2 \, . \, . \, n} + f^{n+1}(\theta x)\frac{x^{n+1}}{1 \, . \, 2 \, . \, . \, . \, . \, (n+1)}, \quad . \quad (1)$$

and

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{1 \, . \, 2} \dots \\ + f^{n-1}(0)\frac{x^{n-1}}{1 \, . \, 2 \, . \, . \, . \, (n-1)} + \left\{ \begin{matrix} f^n(0) \\ f^n(x) \end{matrix} \right\} \frac{h^n}{1 \, . \, 2 \, . \, . \, . \, n} \, . \, . \quad (2)$$

The notation $\left\{ \begin{matrix} f^n(0) \\ f^n(x) \end{matrix} \right\}$ denotes a coefficient of the fraction intermediate in value between $f^n(0)$ and $f^n(x)$ [see Art. 139].

141. This form of development is in reality no less general than that of Taylor; for any function which is included in the general form $f(x_0 + h)$ may also, by giving a different signification to the symbol f, be expressed in the form $f(h)$, or, employing the notation of the last article, $f(x)$. Thus, if $\log(1 + h)$ be developed by Taylor's Theorem $f(x) = \log x$, the value of x_0 being unity; but, if $\log(1 + x)$ is to be developed by Maclaurin's Theorem, we put $f(x) = \log(1 + x)$. (Compare Ex. 1, XXII, with Art. 144.)

THE NAPIERIAN BASE.

142. As an example of the application of the above theorem, we shall deduce the exponential series or development of the function ϵ^x, and shall thence obtain a series for computing the value of ϵ.

The successive derivatives of ϵ^x being equal to the original function, the coefficients, $f(0)$, $f'(0)$, &c., each reduce to unity; we therefore derive, by substituting in Equation (1),

$$\epsilon^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \dots \frac{x^n}{1.2.....n} + \epsilon^{\theta x} . \frac{x^{n+1}}{1.2....(n+1)} .$$

Putting x equal to unity, we obtain the following series, which enables us to compute the value of the incommensurable quantity ϵ to any required degree of accuracy:

$$\epsilon = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} \dots + \frac{1}{1.2.3..n} + \frac{\epsilon^\theta}{1.2.3..(n+1)} .$$

The computation may be arranged thus:

$$\begin{array}{r}
2.5 \\
.16666666667 \\
4166666667 \\
833333333 \\
138888889 \\
19841270 \\
2480159 \\
275573 \\
27557 \\
2505 \\
209 \\
16 \\
1 \\
\hline
2.71828182846
\end{array}$$

Since ϵ^θ is less than ϵ, the remainder (n being 14) is less than $\frac{3}{15}$ of the last term employed in the computation, and therefore cannot affect the result. Inasmuch as the last figure of each term may contain a positive or negative error of one-half, we cannot, in general, rely upon the accuracy of the last two places of decimals in computations involving so large a number of terms. Accordingly, this computation only justifies us in writing

$$\epsilon = 2.718281828.$$

It happens, however, that in the above computation the result is correct to the last place of decimals.

SYMBOLIC FORM OF TAYLOR'S THEOREM.

143. By employing the notation $\frac{d}{dx}, \frac{d^2}{dx^2}$, &c., for the derivatives, we may write Taylor's Theorem in the form

$$f(x+h)=f(x)+h\frac{d}{dx}f(x)+\frac{h^2}{1.2}\cdot\frac{d^2}{dx^2}f(x)+\frac{h^3}{1.2.3}\cdot\frac{d^3}{dx^3}f(x)+\&c.$$

The form

$$f(x+h)=\left\{1+h\frac{d}{dx}+\frac{h^2}{1.2}\cdot\frac{d^2}{dx^2}+\frac{h^3}{1.2.3}\cdot\frac{d^3}{dx^3}+\ldots\&c.\right\}f(x),$$

in which $f(x)$ is removed from within the braces as though it were a factor, may also be employed, inasmuch as this form of writing the second member admits of no ambiguity of meaning. It will be noticed that the terms within the braces proceed according to the law of the exponential series (Art. 142); we may therefore write

$$f(x+h)=\varepsilon^{h\frac{d}{dx}}f(x),$$

which is to be understood as implying that $\varepsilon^{h\frac{d}{dx}}$ is to be developed in accordance with the form of the exponential series, and $f(x)$ introduced after each term.

LOGARITHMIC SERIES.

144. The logarithmic series is deduced by applying Maclaurin's Theorem to the function $\log(1+x)$.

In this case

$$f(x)=\log(1+x) \quad \therefore \quad f(0)=0$$

$$f'(x)=\frac{1}{1+x} \quad \therefore \quad f'(0)=1$$

$$f''(x)=-\frac{1}{(1+x)^2} \quad \therefore \quad f''(0)=-1$$

$$f'''(x)=\frac{1.2}{(1+x)^3} \quad \therefore \quad f'''(0)=1.2$$

$$f^{\text{iv}}(x)=+\frac{1.2.3}{(1+x)^4} \quad \therefore \quad f^{\text{iv}}(0)=-1.2.3$$

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\&c. \qquad . \quad (1)$$

Since this series is divergent for values of x greater than unity, we proceed to deduce a formula for the difference of two logarithms, which may be employed in computing successive logarithms.

145. Denoting the numbers corresponding to two logarithms by n and $n+h$, we derive a series for

$$\log(n+h) - \log n, \quad \text{or} \quad \log\frac{n+h}{n}.$$

A series which could be employed for this purpose might be obtained from (1) by putting $\frac{n+h}{n}$ in the form $1+\frac{h}{n}$. We obtain, however, a much more convergent series by the process given below.

Substituting $-x$ for x in (1), we have

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \&c. \quad . \quad (2)$$

Subtracting (2) from (1),

$$\log\frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \&c.\right]; \quad . \quad (3)$$

a series involving only the positive terms of series (1).

Putting $\frac{1+x}{1-x} = \frac{n+h}{n}$, we derive $x = \frac{h}{2n+h}$; substituting in (3), we have

$$\log\frac{n+h}{n} = 2\left[\frac{h}{2n+h} + \tfrac{1}{3}\frac{h^3}{(2n+h)^3} + \tfrac{1}{5}\frac{h^5}{(2n+h)^5} + \&c.\right] \quad . \quad (4)$$

THE COMPUTATION OF THE MODULUS OF THE TABULAR LOGARITHMS.

146. The above series enables us to compute Napierian logarithms. To obtain the common or tabular logarithms, of which 10 is the base, we make use of the relation

$$\log_\epsilon x = \log_\epsilon 10 \log_{10} x,$$

whence

$$\log_{10} x = \frac{1}{\log_\epsilon 10} \cdot \log_\epsilon x.$$

The quantity $\frac{1}{\log_\epsilon 10}$ is called the modulus of common logarithms, and is denoted by M.

The approximate numerical value of $\log_\epsilon 10$ could be computed by putting $n = 1$ and $h = 9$ in (4); but, since the series thus obtained would converge very slowly, it is more convenient first to compute log 2 by means of the series obtained by putting $n = 1$ and $h = 1$ in (4); thus:

$$\log_\epsilon 2 = 2\left[\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \frac{1}{7}\cdot\frac{1}{3^7} + \&c.\right].$$

We then put $n = 8$ and $h = 2$ in (4); whence, putting the factor $\frac{1}{3}$ outside the brackets, we have

$$\log_\epsilon 10 = 3\log_\epsilon 2 + \frac{2}{3}\cdot\left[\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^5} + \frac{1}{5}\cdot\frac{1}{3^9} + \frac{1}{7}\cdot\frac{1}{3^{13}} + \&c.\right].$$

The numerical work may be arranged thus:

$\frac{1}{3}$	0.3333333333	:	1	0.3333333333
$(\frac{1}{3})^3$	370370370	:	3	123456790
$(\frac{1}{3})^5$	41152263	:	5	8230453
$(\frac{1}{3})^7$	4572474	:	7	653211
$(\frac{1}{3})^9$	508053	:	9	56450
$(\frac{1}{3})^{11}$	56450	:	11	5132
$(\frac{1}{3})^{13}$	6272	:	13	482
$(\frac{1}{3})^{15}$	697	:	15	46
$(\frac{1}{3})^{17}$	77	:	17	5
				0.3465735902
				2
			$\log_\epsilon 2 =$	0.6931471804

$\frac{1}{3}$	0.3333333333	:	1	0.3333333333
$(\frac{1}{3})^5$	41152263	:	3	13717421
$(\frac{1}{3})^9$	508053	:	5	101611
$(\frac{1}{3})^{13}$	6272	:	7	896
$(\frac{1}{3})^{17}$	77	:	9	9
				0.3347153270
				0.1115717757
				0.2231435513
			$3\log_\epsilon 2 =$	2.0794415412
			$\log_\epsilon 10 =$	2.30258509

$$M = \frac{1}{\log_\epsilon 10} = 0.43429448$$

THE DEVELOPMENT OF IMPLICIT FUNCTIONS.

147. When Maclaurin's Theorem is used to develop implicit functions, it is convenient to employ the following method of writing the series:

Let y denote the function, and y_0 its value when x is zero; then $\left.\frac{dy}{dx}\right]_0$, $\left.\frac{d^2y}{dx^2}\right]_0$, &c., will denote the corresponding values of the derivatives, hence the equation may be written thus:

$$y = y_0 + \left.\frac{dy}{dx}\right]_0 x + \left.\frac{d^2y}{dx^2}\right]_0 \frac{x^2}{1.2} + \&c.$$

The process of evaluating the derivatives in the case of implicit functions is exemplified below. A few terms only of the development are usually found in this way, as the process is necessarily tedious.

Let $y^3 - 6xy - 8 = 0 \quad \therefore \quad y_0 = 2$ (the only real value of y_0)

$$(y^2 - 2x)\frac{dy}{dx} - 2y = 0 \quad \therefore \quad \left.\frac{dy}{dx}\right]_0 = 1.$$

$$2y\left(\frac{dy}{dx}\right)^2 - 4\frac{dy}{dx} + (y^2 - 2x)\frac{d^2y}{dx^2} = 0 \quad \therefore \quad \left.\frac{d^2y}{dx^2}\right]_0 = 0.$$

In a similar manner, we obtain

$$\left.\frac{d^3y}{dx^3}\right]_0 = -\tfrac{1}{2}, \quad \text{and} \quad \left.\frac{d^4y}{dx^4}\right]_0 = 1.$$

Whence

$$y = 2 + x - \tfrac{1}{2}\cdot\frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c.$$

EXAMPLES XXIII.

1. Expand $\sin x$ in powers of x.

Employing the expression for the nth derivative [see Ex. 9, XIV],

$$f^n(x) = \sin(x + \tfrac{1}{2}n\pi) \quad \therefore \quad f^n(0) = \sin(\tfrac{1}{2}n\pi).$$

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2\ldots5} - \ldots$$
$$+ \frac{x^n}{1.2\ldots n}\sin(\tfrac{1}{2}n\pi) + \frac{x^{n+1}}{1.2\ldots(n+1)}\sin[\tfrac{1}{2}(n+1)\pi + \theta x].$$

2. Expand $\cos x$ in powers of x.

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \dots$$

$$+ \frac{x^n}{1.2\dots n}\cos\left(\tfrac{1}{2}n\pi\right) + \frac{x^{n+1}}{1.2\dots(n+1)}\cos\left[\tfrac{1}{2}(n+1)\pi + \theta x\right].$$

3. Expand $(1+x)^m$.

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + \text{\&c.}$$

It is evident that no coefficient will vanish if m is negative or fractional. This is the form in which the binomial theorem is employed in computation, x being less than unity.

4. Expand $x\varepsilon^x$, making use of the nth derivative [see Ex. 14, XIV], to find each term. Verify by multiplying the expanded value of ε^x by x.

5. Write the expansion of $x^2\varepsilon^x$, and verify by means of the expression for the nth derivative [see Ex. 15, XIV].

6. From the expanded value of ε^x, deduce the expansion of $x\varepsilon^{2x}$, and verify by means of the nth derivative [see Ex. 16, XIV].

7. Derive the expansion of $\log(1-x^2)$ from the logarithmic series, and verify by adding the expansions of $\log(1+x)$ and $\log(1-x)$.

8. Expand $\log(1-x+x^2)$.

If we put $1-x+x^2$ in the form $\dfrac{1+x^3}{1+x}$, the logarithmic series may be employed:

$$\log(1-x+x^2) = -x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{3} - \text{\&c.}$$

9. Expand $\varepsilon^x \cos x$.

$$\varepsilon^x \cos x = 1 + x - \frac{2x^3}{1.2.3} - \frac{2^2x^4}{1.2\dots4} - \frac{2^2x^5}{1.2\dots5}$$

$$+ \frac{2^3x^7}{1.2\dots7} + \frac{2^4x^8}{1.2\dots8} + \text{\&c.}$$

10. Expand $(1+\varepsilon^x)^n$.

$$(1+\varepsilon^x)^n = 2^n\left\{1 + \frac{n}{2}.x + \frac{n(n-1)}{2^2}.\frac{x^2}{1.2}\right.$$

$$\left. + \frac{n^2(n+3)}{2^3}.\frac{x^3}{1.2.3} + \text{\&c.}\right\}.$$

11. Expand $\sin^2 x$.

$$\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2x^6}{3^2 . 5} - \,.\,.\ \&c.$$

12. Derive the expansion of $(1 + x)\varepsilon^x$ from that of ε^x.

13. Expand $\log(1 + \sin x)$ to the term involving x^4 inclusive.

$$\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \,.\,.\ \&c.$$

14. Expand $\tan x$ to the term involving x^5 inclusive.

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \,.\,.\ \&c.$$

15. Expand $\sec x$ to the term involving x^6 inclusive.

$$\sec x = 1 + \frac{x^2}{1 . 2} + \frac{5x^4}{1 . 2 . 3 . 4} + \frac{61x^6}{1 . 2 \ldots . 6} + \,.\,.\ \&c.$$

16. Expand $\log \sec x$ to the term involving x^6 inclusive.

$$\log \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \,.\,.\ \&c.$$

17. Expand $\varepsilon^x \sec x$.

$$\varepsilon^x \sec x = 1 + x + x^2 + \frac{2x^3}{3} + \,.\,.\ \&c.$$

18. Find the expansion of $\varepsilon^x \log(1 + x)$ to the term involving x^5 by multiplying together a sufficient number of the terms of the series for ε^x and for $\log(1 + x)$.

$$\varepsilon^x \log(1 + x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \,.\,.\ \&c.$$

19. Expand $\log(1 + \varepsilon^x)$.

$$\log(1 + \varepsilon^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \,.\,.\ \&c.$$

20. Compute $\log_\varepsilon 3$, and find $\log_{10} 3$ by multiplying by the value of M [Art. 146].

$$\log_\varepsilon 3 = 1.0986123.$$
$$\log_{10} 3 = 0.4771213.$$

21. Find $\log_\varepsilon 269$.

Put $n = 270 = 10 \times 3^3$, and $h = -1$.

$$\log_\varepsilon 269 = 5.5947114.$$

22. Given $\varepsilon^y + xy = \varepsilon$, to find the expansion of y in powers of x.

$$y = 1 - \frac{x}{\varepsilon} + \frac{1}{1 . 2} \cdot \frac{x^2}{\varepsilon^2} + \frac{1}{1 . 2 . 3} \cdot \frac{x^3}{\varepsilon^3} + \,.\,.\ \&c.$$

23. Given $y^2 - xy = 1$, to find the expansion of y in powers of x.

$$y = \pm 1 + \tfrac{1}{2} . x \pm \frac{1}{2^2} \cdot \frac{x^2}{1 . 2} \mp \frac{3}{2^4} \cdot \frac{x^4}{1 . 2 . 3 . 4} \pm \ . \ . \ \&c.$$

24. Given $y = 1 + x\varepsilon^y$, to find the expansion of y in powers of x.

$$y = 1 + \varepsilon x + 2\varepsilon^2 . \frac{x^2}{1 . 2} + 9\varepsilon^3 . \frac{x^3}{1 . 2 . 3} + \ . \ . \ \&c.$$

XXIV.

EVEN FUNCTIONS AND ODD FUNCTIONS.

148. Many simple functions, like $\cos x$, $\log(1+x^2)$, and $a+x^2+bx^4$, have this property, viz:

$$f(x)=f(-x).$$

Every rational integral function involving powers of x with even exponents only has, evidently, this property; and it is also evident that the series corresponding to any function having this property can consist only of such terms. These functions are, therefore, called *even functions.*

Certain other functions, like $\sin x$, $\tan x$, and $ax+bx^3$, have this property, viz:

$$f(x)=-f(-x).$$

It is evident that the series corresponding to a function having this property will contain the odd powers of x only. Such functions are called *odd functions.*

Since, in the case of an odd function,

$$f(0)=-f(0),$$

we must have

$$f(0)=0.$$

Since developments usually contain both odd and even powers, the number of functions included in these two classes is not great.

149. *When $f(x)$ denotes an even function of x, the curve*

$$y=f(x)$$

will be symmetrical with respect to the axis of y, for let

$$b=f(a)=f(-a);$$

then the point (a, b) is on the curve, and the point $(-a, b)$, symmetrical with it, is also on the curve.

If however $f(x)$ denote an odd function of x, the curve will be symmetrical with respect to the origin as a centre; for, in this case, the point (a, b) being on the curve, the point $(-a, -b)$, symmetrical to it with respect to the origin, is also on the curve.

The derivative of an even function is an odd function; for, if we take the derivative of

$$f(x) = f(-x),$$

we obtain

$$f'(x) = -f'(-x).$$

In like manner, it may be proved that *the derivative of an odd function is an even function.*

DE MOIVRE'S THEOREM.

150. Functions of imaginary quantities are, in general, imaginary quantities capable of expression in the form $a + b\sqrt{-1}$. The transformation of a function to this form may often be effected by developing it into a series, and separating the real and imaginary parts.

We shall use the notation i for $\sqrt{-1}$; whence

$$i = \sqrt{-1},\ i^2 = -1,\ i^3 = -\sqrt{-1},\ i^4 = 1, \text{ \&c.}$$

Thus, to transform the imaginary function ϵ^{ix}, we put ix for x in the exponential series, and obtain

$$\epsilon^{ix} = 1 + ix - \frac{x^2}{1.2} - \frac{ix^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \text{\&c.},$$

or

$$\epsilon^{ix} = \left[1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \text{\&c.}\right] + i\left[x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \text{\&c.}\right].$$

The series in brackets are the developments of $\cos x$ and $\sin x$ respectively [see Ex. 2 and Ex. 1, XXIII]; we have therefore, by substitution,

$$\epsilon^{ix} = \cos x + i \sin x. \quad . \quad , \quad . \quad (1)$$

Putting mx for x, we have also,

$$\epsilon^{imx} = \cos mx + i \sin mx; \quad . \quad . \quad . \quad (2)$$

and, since

$$\epsilon^{imx} = (\epsilon^{ix})^m,$$

we have

$$(\cos x + i \sin x)^m = \cos mx + i \sin mx. \quad . \quad (3)$$

This result is known as De Moivre's Theorem.

151. If we expand the first member by the binomial theorem, the real part will be an expansion of $\cos mx$ in terms of $\sin x$ and $\cos x$, and the coefficient of the imaginary part will be an expansion of $\sin mx$ also in terms of $\sin x$ and $\cos x$. Thus, putting $m = 3$ in Equation (3), we obtain

$$\cos 3x + i \sin 3x = \cos^3 x + 3 \cos^2 x . i \sin x - 3 \cos x . \sin^2 x - i \sin^3 x.$$

Whence

$$\cos 3x = \cos^3 x - 3 \cos x \sin^2 x,$$

and

$$\sin 3x = 3 \cos^2 x \sin x - \sin^3 x.$$

152. Exponential expressions for $\sin x$ and $\cos x$ may be derived from (1); thus, putting $-x$ for x, we have, since $\cos(-x) = \cos x$,

$$\varepsilon^{-ix} = \cos x - i \sin x, \quad . \quad . \quad . \quad (4)$$

Eliminating $\sin x$ by adding Equations (1) and (4),

$$\cos x = \tfrac{1}{2}(\varepsilon^{ix} + \varepsilon^{-ix}),$$

Eliminating $\cos x$,

$$\sin x = \frac{1}{2i}(\varepsilon^{ix} - \varepsilon^{-ix}).$$

153. The following expressions are easily deduced from the above formulas:

$$\cos ix = \tfrac{1}{2}(\varepsilon^{x} + \varepsilon^{-x}),$$
$$\sin ix = \tfrac{1}{2}i(\varepsilon^{x} - \varepsilon^{-x}).$$

The value of $\cos ix$ is real and is called the hyperbolic cosine of x; the real factor of $\sin ix$ is called the hyperbolic sine of x. These real functions are generally denoted by $\cosh x$ and $\sinh x$; also, the ratio $\frac{\sinh x}{\cosh x}$ by $\tanh x$. Thus, we have

$$\cosh x = \tfrac{1}{2}(\varepsilon^{x} + \varepsilon^{-x})$$
$$\sinh x = \tfrac{1}{2}(\varepsilon^{x} - \varepsilon^{-x})$$
$$\tanh x = \frac{\varepsilon^{x} - \varepsilon^{-x}}{\varepsilon^{x} + \varepsilon^{-x}}.$$

The reciprocals of these quantities are denoted by sech x, cosech x, and coth x, respectively.

Formulas involving these functions, and bearing a remarkable analogy to the trigonometric formulas, may readily be deduced. [See Ex. 7, 8, 9, XXIV.]

EMPLOYMENT OF THE DERIVATIVES OF AN ASSUMED SERIES.

154. The application of Maclaurin's Theorem to certain functions is very difficult in consequence of the complication of the process of determining the successive derivatives. In such cases, we may frequently employ with advantage an extension of the method of assuming a series with undetermined coefficients, exemplified in Art. 128. This extension depends upon the employment of the derivatives of the assumed series, as given below. Let y denote the given function of x, and assume

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 \dots \\ + A_nx^n + A_{n+1}x^{n+1} + A_{n+2}x^{n+2} + \&c.: \quad (1)$$

whence we obtain, for the first derivative, the series

$$\frac{dy}{dx} = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 \dots \\ + nA_nx^{n-1} + (n+1)A_{n+1}x^n + (n+2)A_{n+2}x^{n+1} + \&c.; \quad (2)$$

and for the second,

$$\frac{d^2y}{dx^2} = 2A_2 + 2.3.A_3x + 3.4.A_4x^2 \dots\dots + (n-1)nA_nx^{n-2} \\ + n(n+1)A_{n+1}x^{n-1} + (n+1)(n+2)A_{n+2}x^n + \&c. \quad (3)$$

In the example of Art. 128, we have, denoting the function by y, the relation

$$y(1-x) = 1.$$

The process there employed is equivalent to substituting for y in this equation the series assumed in (1). The result is an identity which serves to determine the values of the assumed coefficients.

155. When the primitive equation, involving x and y, is not of the proper form, an equation involving one or more derivatives may be used. Thus, to expand $\tan^{-1} x$ into a series, we put

$$y = \tan^{-1} x;$$

whence

$$\frac{dy}{dx} = \frac{1}{1+x^2},$$

or

$$(1+x^2)\frac{dy}{dx} = 1.$$

Substituting the value of $\frac{dy}{dx}$ from (2), we obtain

$$\begin{array}{l} 1 = A_1 + 2A_2x + \left.\begin{array}{r}3A_3\\+A_1\end{array}\right| x^2 + \left.\begin{array}{r}4A_4\\+2A_2\end{array}\right| x^3 \ldots \\ \qquad + nA_nx^{n-1} + (n+1)A_{n+1}x^n + \left.\begin{array}{r}(n+2)A_{n+2}\\+nA_n\end{array}\right| x^{n+1} + \&c. \end{array}$$

Putting the coefficient of x^{n+1} equal to zero, we have the following general relation between the assumed coefficients:

$$(n+2)A_{n+2} + nA_n = 0,$$

or

$$A_{n+2} = -\frac{n}{n+2}A_n.$$

After determining A_0 and A_1, the remaining coefficients may be determined by means of this general relation between A_{n+2} and A_n. A_0 is of course the value of y when $x = 0$, and A_1 is, by Equation (2), the value of $\left.\frac{dy}{dx}\right]_0$.

Hence,

$$A_0 = 0, \quad \text{and} \quad A_1 = 1.$$

Determining the succeeding coefficients by the above equation, we have

$$\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \&c.$$

156. In certain other examples, although this method is applicable neither to the primitive equation between x and y, nor to any of its derivatives, yet, by combining them, a differential equation may be formed to which the method can be applied. If the differential equation thus obtained is rational and integral, and involves neither powers nor products of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c., the labor of squaring a series, or of finding the product of two series, is avoided, and a general relation between the coefficients is readily deduced.

As an illustration, we deduce the development of $\sin mz$ in powers of $\sin z$.

Let

$$\sin z = x, \quad \therefore \quad z = \sin^{-1}x;$$

then it is proposed to develop

$$y = \sin [mz] = \sin [m \sin^{-1} x]$$

in powers of x. In this case,

$$\frac{dy}{dx} = \frac{m \cos [m \sin^{-1} x]}{\sqrt{1 - x^2}},$$

and

$$\frac{d^2y}{dx^2} = \frac{- m^2 \sin [m \sin^{-1} x] + m \cos [m \sin^{-1} x] \dfrac{x}{\sqrt{1 - x^2}}}{1 - x^2}.$$

Whence,

$$(1 - x^2) \frac{d^2y}{dx^2} = - m^2 y + x \frac{dy}{dx},$$

or

$$m^2 y + \frac{d^2y}{dx^2} - x \frac{dy}{dx} - x^2 \frac{d^2y}{dx^2} = 0.$$

Referring to (1), (2), and (3), we find that the coefficient of x_n in the equation derived by substitution would be

$$m^2 A_n + (n + 2)(n + 1) A_{n+2} - n A_n - n(n - 1) A_n.$$

Putting this expression equal to zero, we have

$$A_{n+2} = \frac{n^2 - m^2}{(n + 1)(n + 2)} A_n.$$

Determining A_0 and A_1, as in the last example, we obtain

$$A_0 = 0, \quad \text{and} \quad A_1 = m;$$

hence, all the coefficients with even subscripts vanish, and, making n equal to 1, 3, 5, &c., we obtain the coefficients of the series:

$$A_3 = m \,.\, \frac{1 - m^2}{1 \,.\, 2 \,.\, 3}, \quad A_5 = m \,.\, \frac{(1 - m^2)(9 - m^2)}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5}. \quad \&\text{c.}$$

Substituting these in the assumed series, and putting for x its value $\sin z$, we have

$$\sin mz = m \sin z \left[1 - \frac{m^2 - 1}{1 \,.\, 2 \,.\, 3} \sin^2 z + \frac{(m^2 - 1)(m^2 - 9)}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} \sin^4 z - \&\text{c.} \right].$$

This series will consist of a finite number of terms when m is an *odd* integer.

In a similar manner, it may be proved that

$$\cos mz = 1 - \frac{m^2 \sin^2 z}{1 \,.\, 2} + \frac{m^2 (m^2 - 4)}{1 \,.\, 2 \,.\, 3 \,.\, 4} \sin^4 z - \&\text{c.}$$

If m be an *even* integer, this series will consist of a finite number of terms.

These two expansions were discovered by Euler, who first published them in 1755.

EXAMPLES XXIV.

1. Prove that each of the following expressions is an odd function of x: $\log\dfrac{1+x}{1-x}$, $\log\tan(\frac{1}{4}\pi + x)$, and $\dfrac{\varepsilon^x+1}{\varepsilon^x-1}$.

2. Prove that the product of two odd functions is an even function, but an odd function of an odd function is an odd function.

3. Find expressions for $\sin 5x$ and $\cos 5x$ by the method of Art. 151.

$$\sin 5x = 5\cos^4 x\sin x - 10\cos^2 x\sin^3 x + \sin^5 x.$$

4. Find expressions for $\sin mx$ and $\cos mx$ by the method of Art. 151.

$$\cos mx = \cos^m x - \frac{m(m-1)}{1.2}\cos^{m-2} x\sin^2 x + \&c.$$

$$\sin mx = m\cos^{m-1} x\sin x - \frac{m(m-1)(m-2)}{1.2.3}\cos^{m-}\ x\sin^3 x + \&c.$$

5. Prove, by Equation 1 of Art. 150, that $\varepsilon^{i\frac{\pi}{2}} = i$, and that $\varepsilon^{i\pi} = -1$; show also that these results are consistent.

6. Prove that $(-1)^{-i}$ is equal to the real quantity ε^π; also, that $\sqrt{\varepsilon^\pi} = \sqrt[i]{i}$.

7. Prove the relations:

$$\cosh^2 x - \sinh^2 x = 1, \quad \text{and} \quad \operatorname{sech}^2 x = 1 - \tanh^2 x.$$

8. Prove the relations:

$$\sinh 2x = 2\sinh x\cosh x, \quad \text{and} \quad \cosh 2x = \cosh^2 x + \sinh^2 x.$$

9. Prove the formulas:

$$\sinh(x \pm y) = \sinh x\cosh y \pm \cosh x\sinh y,$$

and

$$\cosh(x \pm y) = \cosh x\cosh y \pm \sinh x\sinh y.$$

10. Expand $\dfrac{\varepsilon^x}{1+x}$ by division, making use of the exponential series.

$$\frac{\varepsilon^x}{1+x} = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \frac{11x^5}{30} + \&c.$$

11. Expand $\sin^{-1}x$.

$\sin^{-}x$ is an odd function of x, and its derivative, which is necessarily an even function, may be expanded by the binomial theorem in terms of x^2. Hence assume

$$\sin^{-1}x = A_1x + A_3x^3 + A_5x^5 + A_7x^7 + \&c. \quad . \quad (1)$$

Whence

$$\frac{1}{\sqrt{1-x^2}} = A_1 + 3A_3x^2 + 5A_5x^4 + 7A_7x^6 + \&c. \qquad (2)$$

But, by the binomial theorem,

$$(1-x^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}\,.\,x^2 + \frac{1\,.\,3}{2\,.\,4}x^4 + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^6 + \&c. \qquad (3)$$

Comparing the coefficients in (2) and (3), we determine A_1, A_3, A_5, &c. Hence, substituting in (1), we have

$$\sin^{-1}x = x + \tfrac{1}{2}\cdot\tfrac{1}{3}x^3 + \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{5}x^5 + \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{1}{7}x^7 + \&c.$$

12. Deduce the expansion of $\log(1+x)$ by the method employed in deducing the expansion of $\tan^{-1}x$ in Art. 155.

13. Deduce the expansion of e^x by the method of Art. 156, deriving the differential equation by combining the primitive equation and its first derivative.

14. Derive by differentiation three terms of the expansion of $\tan x$ from that of $\log \sec x$ [Ex. 16, XXIII], and from the result derive the expansion of $\sec^2 x$.

15. Show that the differential equation $\frac{d^2y}{dx^2} + y = 0$ can be deduced from any function of the form $y = a\sin x + b\cos x$, and from this differential equation deduce the series for $\sin x$ and that for $\cos x$.

16. Expand $\log[x + \sqrt{(a^2+x^2)}]$ by the method employed in Ex. 11.

$$\log[x + \sqrt{(a^2+x^2)}] = \log a + \frac{x}{a} - \tfrac{1}{2}\,.\,\frac{x^3}{3a^3} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{x^5}{5a^3} - \ldots \&c.$$

17. By means of the expansion of $\cos mz$ (Art. 156) prove that

$$\cos z = 1 - \tfrac{1}{2}\sin^2 z - \tfrac{1}{8}\sin^4 z - \tfrac{1}{16}\sin^6 z - \&c.$$

18. By means of Euler's expansions derive the values of $\cos 4z$ and $\sin 5z$.

19. From the same formula, obtain the expansion for $\cos\frac{1}{2}z$.

20. Expand ϵ^{ϵ^x} to the term containing x^3 inclusive.

Denoting the function by y, we easily deduce $\frac{dy}{dx} = y \, . \, \epsilon^x$; therefore the product of the assumed series and the exponential series must be equivalent to the derivative of the former.

$$\epsilon^{\epsilon^x} = \epsilon \, [1 + x + x^2 + \tfrac{5}{6}x^3 + \text{\&c.}]$$

21. Expand $\epsilon^{m \sin^{-1} x}$ by the method of Art. 156.

$$\epsilon^{m \sin^{-1} x} = 1 + mx + \frac{m^2x^2}{1 \, . \, 2} + \frac{m \, (m^2 + 1)}{1 \, . \, 2 \, . \, 3}x^3 + \frac{m^2 \, (m^2 + 4)}{1 \, . \, 2 \, . \, 3 \, . \, 4}x^4 + \text{\&c.}$$

22. Expand y in powers of x when y satisfies the differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0,$$

and

$$y_0 = a.$$

$$y = a\left[1 - x + \frac{x^2}{4} - \frac{x^3}{4 \, . \, 9} + \frac{x^4}{4 \, . \, 9 \, . \, 16} - \frac{x^5}{4 \, . \, 9 \, . \, 16 \, . \, 25} + \text{\&c.}\right].$$

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